Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Linear Algebra/Advanced Calculus

January 20, 2004

Do 7 of the following 9 problems. Indicate clearly which problems should be graded.

Passing Standard: For Master's level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Part I Linear Algebra

1. Let V be a finite dimensional vector space over $\mathbb{R}$, with fixed but *arbitrary* basis $B = \{v_1, \ldots, v_n\}$. Suppose $\langle u, v\rangle$ is an inner product on V: symmetric, bilinear, positive definite. If $c_1, \ldots, c_n$ are arbitrary scalars, prove that there is a unique $v \in V$ for which $\langle v, v_k\rangle = c_k$ for all k.

2. Let A be a (real) *orthogonal* $n \times n$ matrix: $A^t A = I$ (where A^t denotes the transpose matrix).

   (a) Prove that $\det A = \pm 1$.

   (b) Prove that x and Ax have the same length, for all $x \in \mathbb{R}^n$.

   (c) Prove that the only possible real eigenvalues of A are ± 1.

   (d) If $n = 3$ and $\det A = 1$, prove that 1 is an eigenvalue of A.

3. (a) Show that if A is a diagonalizable matrix with non-negative real eigenvalues, then there is a matrix S such that $S^2 = A$.

   (b) Using the general method from (a), find a matrix whose square is

$$A = \begin{pmatrix} 1 & 0 & 8 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

4. Let $\mathcal{M}$ be the n^2-dimensional space of $n \times n$ matrices over $\mathbb{R}$. Let $\mathcal{S}$ resp. $\mathcal{K}$ be the subspace consisting of *symmetric* matrices A (satisfying $A^t = A$) resp. *skew-symmetric* matrices A (satisfying $A^t = -A$).

   (a) Prove that each $n \times n$ matrix A can be written uniquely as the sum of a symmetric matrix B and a skew-symmetric matrix C. (In other words, $\mathcal{M}$ is the direct sum of $\mathcal{S}$ and $\mathcal{K}$.)

   (b) Compute the dimensions of $\mathcal{S}$ and $\mathcal{K}$.

Part II Advanced Calculus

1. For the radial vector $\vec{r} = (x, y, z)$ and a suitable region R in $\mathbb{R}^3$, show that
$$\iiint\limits_{R} \frac{dV}{r^2} = \iint\limits_{S=\partial R} \frac{\vec{r}}{r^2} \cdot \overrightarrow{dS}$$
for $r = \|\vec{r}\|$.

2. Let S be a smooth surface in $\mathbb{R}^3$ with (oriented) boundary ∂S. If f and g are C^1-functions, show that
$$\int_{\partial S} f\,\nabla g \cdot d\vec{s} = \iint_{S} (\nabla f \times \nabla g) \cdot \overrightarrow{dS}$$

3. Define a sequence of functions $f_n : [0, \pi] \to \mathbb{R}$ by
$$f_n(x) = \begin{cases} \sin(nx) \text{ if } 0 \le x \le \pi/n \\ 0 \text{ otherwise} \end{cases}$$
Determine whether $\{f_n\}$ converges (a) pointwise (b) uniformly.

4. Determine (using just the definitions involved) whether each function is (a) *continuous* at 0, (b) *differentiable* at 0:
$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for } n = 1, 2, \ldots, \\ 0 & \text{otherwise} \end{cases}$$
$$g(x) = \begin{cases} 1/n \text{ if } x = 1/n \text{ for } n = 1, 2, \ldots, \\ 0 \text{ otherwise} \end{cases}$$

5. (a) Determine the radius of convergence R and the interval of convergence for the power series
$$\sum_{n=1}^{\infty} \frac{3^n}{n \cdot 4^n} x^n$$
(b) If a power series $\sum a_n x^n$ has radius of convergence $R > 0$ and if $0 < R' < R$, prove that the series converges uniformly on $[-R', R']$. Conclude that $f(x) = \sum a_n x^n$ defines a *continuous* function on $(-R, R)$.