University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
January 22, 2004

Do 5 out of the following 7 questions. Indicate clearly which questions you want to have graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

Problem 1. Identify, in the usual way, $\mathbb{C}^2 \cong \mathbb{R}^4$ and consider the map

$$h\colon \mathbb{C}^2 \to \mathbb{R}^3 \; ; \quad (u,v) \mapsto (2\,\mathrm{Re}(u\bar{v}), 2\,\mathrm{Im}(u\bar{v}), |u|^2 - |v|^2)\,.$$

Let $S^3 = \{(u,v) \in \mathbb{C}^2 : |u|^2 + |v|^2 = 1\}$.

a) Prove that $h(S^3) \subset S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and show that $H = h|_{S^3}\colon S^3 \to S^2$ is a C^∞ map.
b) Show that $H\colon S^3 \to S^2$ is a surjective submersion.
c) Let $p = (1/\sqrt{3}\,, -1/\sqrt{3}\,, 1/\sqrt{3}\,) \in S^2$. Determine $H^{-1}(p)$.

Problem 2. Let $M = \mathbb{R}^2$ with the Riemannian metric:

$$g(\partial/\partial x, \partial/\partial x) = e^{x+y} \;; \quad g(\partial/\partial y, \partial/\partial y) = e^{x-y} \;; \quad g(\partial/\partial x, \partial/\partial y) = 0.$$

a) Compute the Gaussian curvature of (M, g).
b) Write, explicitly, the differential equations of a geodesic in (M, g).

Problem 3. Let $G = \left\{ \begin{pmatrix} x & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;:\; x, y, z \in \mathbb{R}\,, x \neq 0 \right\}$

a) Prove that G is a Lie subgroup of $GL(3, \mathbb{R})$.
b) Find a basis of left-invariant 1-forms on G.
c) Find a left-invariant Riemannian metric on G. Write your answer in terms of dx, dy, dz.

Problem 4. Let (M, g) be an oriented Riemannian manifold and let X be a vector field on M.

(i) Define the divergence of X.
(ii) Prove that

$$\int_D \mathrm{div}\,(X) = \int_{\partial D} g(X, N)$$

where D is a regular domain in M (i.e., the boundary of D is a smooth hyper surface) and N is the outward unit normal to D.
(iii) Deduce the divergence theorem in $\mathbb{R}^2$ from this.

Problem 5. Show that over the circle S^1 there are exactly two isomorphism classes of rank r real vector bundles.

Problem 6. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be a 2-torus and consider the trivial rank n bundle $V = M \times \mathbb{R}^n$ over M. We equip V with the connection $\nabla = d + Adx + Bdy$ where d denotes the trivial connection given by directional derivatives of $\mathbb{R}^n$-valued functions, A, B are $n \times n$ matrices and dx, dy are the coordinate differentials on $\mathbb{R}^2$. Show:

(i) ∇ is flat if and only if the matrices A, B commute, i.e., $[A, B] = 0$.
(ii) Assuming ∇ to be flat compute its holonomy representation $H : \mathbb{Z}^2 \to \mathbf{Gl}(n, \mathbb{R})$.
(iii) Assuming ∇ to be flat then V admits a non-trivial parallel section if and only if A, B have a common kernel.

Problem 7. Show that any smooth (paracompact, second countable, Hausdorff) manifold M admits a Riemannian metric. What can you say about the non-positive definite case, i.e., pseudo-Riemannian metrics?