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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.

2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.

3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. (a) Let f be a Lebesgue-integrable function on $[0, \infty)$ and let α be a positive real number. Using properties of Lebesgue measure m, prove that

$$\int_0^\infty f(x/\alpha)\, dm(x) \;=\; \alpha \int_0^\infty f(x)\, dm(x).$$

(b) Let g be a Lebesgue-integrable function on $[0, \infty)$ satisfying

$$\int_0^\infty g(x)\, dm(x) \;=\; 1.$$

Let h a bounded, Borel-measurable function satifying $\lim_{x\to\infty} h(x) = L$ for some $L \in I\!R$. Compute the following limit and justify your answer:

$$\lim_{n\to\infty} \frac{1}{n} \int_0^\infty g(x/n)\, h(x)\, dm(x).$$

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \mathcal{N})$ a measurable space. Let $T\colon X \to Y$ be a measurable function.

(a) Define a set function ν on $\mathcal{N}$ by $\nu(A) = \mu(T^{-1}(A))$ for every $A \in \mathcal{N}$. Prove that ν is a measure on $\mathcal{N}$.

(b) Prove that if $f \in L^1(\nu)$, then $f \circ T \in L^1(\mu)$ and that

$$\int_Y f \, d\nu = \int_X (f \circ T) \, d\mu.$$

(c) Consider $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N})$ as in line 1 of this problem and consider the measure ν defined in part (a). Assume that $\mu(X) < \infty$ and that γ is a finite measure on $(Y, \mathcal{N})$ satisfying $\nu \ll \gamma$. By using part (b) and quoting a well known theorem in measure theory, prove that there exists $g \in L^1(\gamma)$ such that

$$\int_X (f \circ T) \, d\mu = \int_Y f \, g \, d\gamma \ \text{ for each } f \in L^1(\nu).$$

3. For $x \in [0, \infty)$ define $f(x) = \int_x^\infty e^{-t^2}\, dm(t)$. Verify that

$$\int_0^\infty f(x)\, dm(x) \;=\; \frac{1}{2},$$

justifying all steps.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_n, n \in I\!N\}$ an arbitrary sequence in $\mathcal{M}$.

(a) Give the definitions of $\liminf E_n$ and $\limsup E_n$ in terms of intersections and unions of E_n.

(b) Define the sets

$$F = \{x \in X : x \text{ lies in all but finitely many } E_n\}$$

and

$$G = \{x \in X : x \text{ lies in infinitely many } E_n\}.$$

Which of the following two pairs of equalities is true?

$$(i)\ \liminf E_n = F\,,\ \limsup E_n = G\,;$$

$$(ii)\ \liminf E_n = G\,,\ \limsup E_n = F\,.$$

Using the definitions in part (a), prove one of the two equalities in the pair that you picked.

(c) Prove that $\liminf_{n\to\infty} 1_{E_n}(x) = 1_{\liminf E_n}(x)$ for all $x \in \mathcal{X}$.

(d) Using a well known limit theorem for integrals, prove that

$$\mu(\liminf E_n) \leq \liminf_{n\to\infty} \mu(E_n).$$

5. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let E be a subset of H. We denote by $\bar{E}$ the smallest closed subspace of $\mathcal{H}$ containing E. We define the set

$$E^{\perp} \;=\; \{u \in \mathcal{H} : \langle u, x\rangle = 0 \;\text{ for all } x \in E\}.$$

(a) Prove that $E^{\perp}$ is a closed subspace of $\mathcal{H}$.

(b) Prove that $E \subset (E^{\perp})^{\perp}$ and that $\bar{E}^{\perp} \subset E^{\perp}$.

(c) For any closed subspace M of $\mathcal{H}$ the following is true: $(M^{\perp})^{\perp} = M$. Using this fact and part (b), prove that $(E^{\perp})^{\perp} = \bar{E}$.

6. Let f be a Lebesgue-integrable function on $\mathbb{R}$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\, \cos(nx)\, dm(x) \;=\; 0.$$

Hint. First verify the assertion for $f = 1_{(a,b)}$, where $-\infty < a < b < \infty$. Then show how the general case follows from this.

7. (a) Let $(X, \mathcal{M})$ be a measurable space and let μ be a nonnegative, finitely additive set function on $\mathcal{M}$ satisfying $\mu(\emptyset) = 0$ and $\mu(X) < \infty$. In addition, assume that μ is continuous from above at the empty set $\emptyset$; that is, if $\{E_j, j \in \mathbb{N}\}$ is any sequence in $\mathcal{M}$ satisfying $E_{j+1} \subset E_j$ for every $j \in \mathbb{N}$ and $\cap_{j \in \mathbb{N}} E_j = \emptyset$, then

$$\lim_{j \to \infty} \mu(E_j) \;=\; \mu(\emptyset) \;=\; 0.$$

Prove that μ is countably additive on $\mathcal{M}$.

(b) Let $\{a_j, j \in \mathbb{N}\}$ be a sequence of nonnegative real numbers for which $\sum_{j \in \mathbb{N}} a_j < \infty$. Define the set function μ on $\mathcal{B}_{\mathbb{R}}$ as follows:

$$\mu(A) \;=\; \sum_{a_j \in A} a_j \;\text{ for every } A \in \mathcal{B}_{\mathbb{R}}.$$

Prove the following: (i) $\mu(\mathbb{R}) < \infty$; (ii) μ is a nonnegative, finitely additive set function on $\mathcal{B}_{\mathbb{R}}$; (iii) μ is continuous from above at the empty set $\emptyset$. Conclude that μ is countably additive on $\mathcal{B}_{\mathbb{R}}$.

8. Let $(X, \mathcal{M}, \rho)$ be a measure space and $f\colon X \to I\!R$ a Borel-measurable function; assume that $f \neq 0$ ρ-a.e. For $u \in [0, \infty)$ define

$$\varphi(u) \;=\; \int_X |f|^u \, d\rho.$$

(a) Let $0 \leq r < s < t$ be real numbers for which $\varphi(r) < \infty$ and $\varphi(t) < \infty$. Prove that $\varphi(s) < \infty$. **Hint.** There exists $\alpha \in (0, 1)$ such that $s = \alpha r + (1 - \alpha)t$.

(b) Assume that $\varphi(u) < \infty$ for all $u \in [0, \infty)$. Prove that $\log \varphi(u)$ is a convex function of $u \in [0, \infty)$.