Department of Mathematics and Statistics
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Basic Exam: Linear Algebra/Advanced Calculus

January 21, 2003

The number of problems. **Do 7 out of the following 9 problems. Indicate clearly which problems should be graded.**

Passing standard. To pass at the Master's level it is sufficient to have 60% with three problems essentially complete (including at least one from each part). To pass at the Ph.D. level, 75% with two questions from each part essentially complete.

Part 1. Linear algebra

(1) On $\mathbb{R}^2$ we denote by $\mathcal{A}_\theta$ the operation of rotation by the angle θ. (A positive angle corresponds to counterclockwise rotation.)
  (a) Write the matrix A_θ of the operation $\mathcal{A}_\theta$ in the standard basis.
  (b) For two angles α and β, derive the formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ by writing the matrix $A_{\alpha+\beta}$ of the rotation $\mathcal{A}_{\alpha+\beta}$, in two different ways.

(2) Let V be a two dimensional vector space over the field of complex numbers. Let T be a linear transformation of V such that $T^2 = 0$ but $T \neq 0$.
  (a) Show that image$(T) \subseteq$ kernel(T).
  (b) Show that there is a basis of V such that the matrix of T in this basis is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
  (c) Do the claims (a) and (b) remain true if V is a two dimensional space over an arbitrary field F?

(3) Let A be an $n \times n$ matrix with complex entries such that $A^k = I_n$ for some positive integer k (here I_n is the $n \times n$ identity matrix). Show that the trace of A satisfies

$$|tr(A)| \leq n.$$

Here $|\cdot|$ is the usual absolute value for complex numbers.

(4) Let V be a finite dimensional vector space over $\mathbb{R}$, equipped with an inner product $\langle -, - \rangle$. Let T be a linear operator on V which is *self adjoint*, i.e.,

$$\langle Tu, v \rangle = \langle u, Tv \rangle \quad \text{for any vectors } u, v \in V.$$

Prove or disprove the following statements, making sure that you justify your answers:
  (a) For any basis $\mathcal{E} = \{e_1, ..., e_n\}$ of V, the matrix $T_\mathcal{E}$ of T in the basis $\mathcal{E}$, is symmetric.
  (b) If v_1 and v_2 are eigenvectors of V corresponding to *different* eigenvalues $\lambda_1 \neq \lambda_2$, then v_1 and v_2 are orthogonal.

Part 2. Advanced Calculus

(1) Compute

$$\int_0^2 \int_{y^2}^4 y\cos(x^2)\ dxdy.$$

(2) Find real numbers A and B such that

$$\lim_{x\to 0} \frac{A\sin(x) - x(1+B\cos(x))}{x^3} = 1.$$

(3) Use the divergence theorem (also called Gauss's theorem) to compute the integral of the normal component of a vector field over a closed surface

$$\int\int_S \big(xy\cdot\boldsymbol{i} + (y^2 + e^{xz^2})\cdot\boldsymbol{j} + \sin(xy)\cdot\boldsymbol{k}\big) \cdot \overrightarrow{dS}.$$

Here, a closed surface S is the boundary of a region bounded by the following four surfaces:

- (i) the xz-plane,

from bellow by

- (ii) the xy-plane,

and from above by both

- (iii) the parabolic cylinder $z = 1 - x^2$, and
- (iv) the plane $z = 2 - y$.

(4) Compute the volume of the region bounded by the paraboloids $z = 9 - x^2 - y^2$ and $z = 3x^2 + 3y^2 - 16$.

(5) Let f be a continuous function on $[0, 1]$ such that

$$f(0) = 0, \quad f(\frac{1}{2}) = 1, \quad f(1) = 0.$$

Define a sequence of functions $f_n(x) = f(x^n)$, $n = 1, 2, 3, ...$ Prove or disprove each of the following statements:

(a) f_n converges pointwise.

(b) f_n converges uniformly on $[0, 1]$.