Department of Mathematics and Statistics
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ADVANCED EXAM — DIFFERENTIAL EQUATIONS
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Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

(1) Consider the system

$$\begin{aligned} x' &= (x^2+y^2-1)x-(x^2+y^2)y \\ y' &= (x^2+y^2)x-(x^2+y^2-1)y. \end{aligned} \tag{1}$$

(a) Show that every solution of (1) satisfies an estimate of the form $x(t)^2 + y(t)^2 \le M$ for some $M > 0$ (depending on the solution).
(b) Determine whether the rest point $(0,0)$ is stable. Is the rest point asymptotically stable? (Justify your answer in each case with a calculation).
(c) How does the solution through a point $(x_o, y_o) \neq (0,0)$ behave as $t \to -\infty$?

(2) Solve the linearized shallow water equations,

$$\begin{pmatrix} u \\ \varphi \end{pmatrix}_t + \begin{pmatrix} \overline{u} & 1 \\ \overline{\varphi} & \overline{u} \end{pmatrix} \begin{pmatrix} u \\ \varphi \end{pmatrix}_x = 0,$$

where $\overline{u}, \overline{\varphi} > 0$ are constants, and with initial data

$$\begin{aligned} u(x,0) &= u_0(x) \\ \varphi(x,0) &= \varphi_0(x). \end{aligned}$$

(3) Consider the minimization problem

$$\inf_{w \in H_0^1(\Omega)} E(w) \tag{2}$$

where

$$E(w) = \int_\Omega \frac{1}{2} |\bigtriangledown w|^2 - wf \; dx$$

and $f \in L^2(\Omega)$ is given, $\Omega \subset \mathbb{R}^d$ connected, bounded with smooth boundary.

(a) Show that a minimizer of (2) is a weak solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

(b) Show that if u is a weak solution of (3) then u is a minimizer of (2).
(c) Show that (2) has a unique minimizer.

(4) (a) Suppose that $0 \leq \alpha < 1$. Calculate an Energy function $E(u, v)$ such that for each solution $(u(t), v(t))$ of (4)

$$\begin{aligned} u' &= v \\ v' &= -\alpha u + (u^2 - 1)(u - \alpha) \end{aligned} \tag{4}$$

$E(u(t), v(t))$ is constant. Use the energy function to sketch the solution curves for each α in the range $0 \leq \alpha < 1$. Use the behavior of solutions in a neighbohood of the point $(u, v) = (\alpha, 0)$ to classify the different phase planes into qualitatively similar groups.
(b) Now consider the system

$$\begin{aligned} u' &= v \\ v' &= -\alpha u + (u^2 - 1)(u - \alpha) \\ \alpha' &= - 100\alpha \end{aligned} \tag{5}$$

Find the *possible* ω limit sets of an orbit through a point (u_o, v_o, α_o).

(5) Prove that if $u \in H^s(\mathbb{R}^2)$ and $s > 1$, then

$$\max_{x \in \mathbb{R}^2} |u(x)| \leq C||u||_s$$

where C is independent of u.

(6)(a) Prove that the largest eigenvalue λ_1 of the operator $Lu = \Delta u + a(x)u$ on the space $H_0^1(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, satisfies the equality

$$\lambda_1 = \max_{u \in H_0^1(\Omega)} Q[u]$$

where $a(x)$ is a given continuous function on $\bar{\Omega}$ and

$$Q[u] = \frac{\int_\Omega (-|\nabla u(x)|^2 + a(x)u(x)^2)\ dx}{\int_\Omega u(x)^2\ dx}.$$

(b) Suppose that n=1 and that $a(x) = \frac{2}{\pi} \arctan\ x$. Find $L_* > 0$ and find a function $u_* \in H_0^1(-L_*, L_*)$ so that $Q[u_*] > 0$ when $L = L_*$.

(7) Consider the system

$$\begin{cases} w_t = w_{xx} + z - w & , \quad x \in (a,b) \quad , \quad t > 0 \\ z_t = z_{xx} + w - z & , \quad x \in (a,b) \quad , \quad t > 0 \\ w(a,t) = w(b,t) = z(a,t) = z(b,t) = 0 & , \quad t \geq 0 \end{cases} \tag{6}$$

(a) Construct a Lyapunov functional of (6) for smooth solutions w and z.
(b) Show that (6) has at most one smooth solution.
(c) Using the Poincaré inequality show that the solution (w, z) of (6) decays exponentially as $t \to \infty$, in a suitable norm.