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ADVANCED EXAM — DIFFERENTIAL EQUATIONS

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Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

1. Let A be the matrix
$$A = \begin{pmatrix} 0 & 1 \\ -13 & -4 \end{pmatrix}.$$

(a) Find the general solution $x(t; y)$ of the initial value problem $x' = Ax$, with $x(0; y) = y$, by calculating the exponential matrix e^{tA}.

(b) Find the ω-limit set $\omega(S)$ of the set
$$S = \{y = \begin{pmatrix} y_1 \\ 0 \end{pmatrix} : y_1 \geq 0\}$$
for the flow $x(t; y)$. Justify your answer with either an argument or a calculation.

2. Solve in an explicit form, using an infinite series, the following initial-boundary-value problem:
$$u_t = ku_{xx} - hu, \qquad 0 < x < \pi, \quad t > 0,$$
$$u(0, t) = u_0, \quad u(\pi, t) = u_1,$$
$$u(x, t) = \phi(x),$$
where k and h are positive constants, the boundary data u_0 and u_1 are given constants, and the initial data ϕ is a smooth function with $\phi(0) = u_0$, $\phi(\pi) = u_1$.

HINT: Decompose the solution into $u = w(x) + v(x, t)$.

3. Consider the system
$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x(x-1)(x-4) + cy, \end{aligned}$$
where c is a parameter.

(a) When $c = 0$, find an energy function which is conserved on solutions of this system. Use this to sketch its global phase plane, including periodic solutions and connecting solutions between rest points, if any. Justify your picture with a thorough mathematical argument showing how it was obtained.

(b) Show that for *some* $c > 0$ there is a solution of the above system running from the rest point $(0, 0)$ as $t \to -\infty$ to $(4, 0)$ as $t \to +\infty$. HINT: study $W^u(0, 0)$ for small and large c.

4. (a) Derive the weak formulation of the elliptic boundary-value problem

$$-\Delta u + cu = f(x) \qquad \text{in } \Omega,$$

$$\frac{\partial u}{\partial \nu} + au = 0 \qquad \text{on } \partial\Omega.$$

Assume that Ω is a bounded domain in R^n, whose boundary $\partial\Omega$ is smooth and has outward directed unit normal ν, that $f \in L^2(\Omega)$, and that the constants a and c are positive.

HINT: First motivate the weak formulation by manipulating classical solutions, and then state it precisely for weakly differentiable functions.

(b) Prove that weak solutions to this BVP are unique.

5. Show by example that the initial-value problem for the Laplace equation is *ill-posed*. Namely, consider the problem

$$u_{xx} + u_{yy} = 0 \qquad 0 < x < 1,\ 0 < y,$$

$$u(x,0) = \phi(x), \quad u_y(x,0) = \psi(x)\,,$$

and show that its smooth solutions have the following property: for every $\eta > 0$, and for any (arbitrarily small) $\delta, \epsilon > 0$, it is possible to find initial data such that

$$\max_{0\le x\le 1} |\phi(x)| \le \delta, \qquad \max_{0\le x\le 1} |\psi(x)| \le \delta,$$

and yet such that the corresponding solution satisfies

$$\max_{0\le x\le 1} |u(x,\eta)| \ge \frac{1}{\epsilon}\,.$$

6. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= -x^3, \quad x(0) = x_0\\ \frac{dy}{dt} &= y + x^2, \quad y(0) = y_0.\end{aligned}$$

(a) Find an explicit (integral) representation for the center manifold $y_o = h(x_0)$ in a neighborhood of the rest point $(0,0)$ for this system.
HINT: Look for solutions $y(t)$ of the second equation which remain bounded as $t \to +\infty$, using the explicit solution of the first equation.
(b) Calculate the first two nonvanishing terms in the Taylor series expansion for $y_0 = h(x_0)$ around $x_0 = 0$.

7. Give an integral formula for the solution $u(x,t)$ of Laplace's equation,

$$\Delta u = 0, \quad \text{in } U = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\},$$

with boundary data

$$u(x,0) = f(x), \qquad u(0,y) = g(y),$$

by constructing the appropriate Green's function.

8. Solve the initial value problem

$$u_{tt} = A\, u_{xx}, \qquad (x,t) \in \mathbb{R}^2, t > 0,$$

with initial data

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x),$$

where u, f and g are vectors in $\mathbb{R}^2$ and A is a symmetric positive definite 2×2 matrix.

HINT: Consider the eigenvectors of A.