DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - NUMERICS
January 24, 2000

Do five of the following problems. All problems carry equal weight.
Unless you note otherwise, only the first five problems will be graded.
Passing level:
Masters: 60% with at least two substantially correct.
Ph.D.: 75% with at least three substantially correct.

1. Determine values of α in the matrix below

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 5 & 8 \\ 4 & 8 & \alpha \end{pmatrix}$$

for which the matrix is positive definite. [Hint: Use Cholesky decomposition.]

2. (a) Write down the Jacobi and the Gauss-Seidel methods for the system $Ax = b$ where

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

(b) Prove that the Jacobi method will converge for any initial guess.

3. Consider the initial value problem

$$y' = f(x, y), \qquad y(0) = y_0,$$

which is discretized by the difference scheme

$$y_{n+1} = 4y_n - 3y_{n-1} - 2hf(x_{n-1}, y_{n-1}).$$

Find the local truncation error for this method.

4. We want to approximate the weighted integral

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}dx$$

by a rule of the form

$$af(0),$$

for general $f(x)$. Assuming that f is sufficiently smooth and that the integral converges,

(a) Find a.

(b) Find the error term. [Hint: You may want to use the formula

$$f[0,x] = f[0,x_1] + (x - x_1)f[0,x_1,x]$$

where x_1 is arbitrary.]

5. We wish to find the smallest positive root of the function

$$f(x) = 1 + \cos x.$$

(a) What is the order of convergence of Newton's method for this function?

(b) To accelerate convergence, we use a modified Newton's method,

$$x_{k+1} = x_k - C\frac{f(x_k)}{f'(x_k)}.$$

Find the value of the constant C for which this iteration converges fastest, and find the order of convergence.

6. A quadratic spline is a continuously differentiable (C^1) piecewise quadratic function. Consider finding the interpolating quadratic spline through the node points

$$(t_0, y_0), (t_1, y_1), \ldots, (t_n, y_n).$$

How many unknown coefficients are there in this problem? How many conditions are given on these unknowns? Derive the equations for the interpolating quadratic spline.

7. Suppose that one wishes to numerically solve the following initial value problem using the Euler method:

$$y' = -5y, \qquad y(0) = 0.$$

However, it is only computationally possible to solve the perturbed problem

$$y_\epsilon' = -5y_\epsilon, \qquad y_\epsilon(0) = \epsilon$$

where ϵ is some small, fixed number. For what choices of step size h will the numerical solution of the perturbed problem converge to the exact solution of the unperturbed problem, 0?