University Of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
January 18, 2000

Do 5 out of the following 7 questions. Indicate clearly which questions you want to have graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

In the problems below, M and N are smooth manifolds.

1. Let $F : M \to N$ be smooth. Prove or disprove:

   a. If $(DF)_p$ is one-to-one $\forall p \in M$, then F is one-to-one.

   b. If $(DF)_p$ is one-to-one for some p, then F is one-to-one in a neighborhood of p.

2. Prove that the tangent bundle of any manifold is, as a manifold, orientable.

3. Prove that the set of points in the integral curve through the identity of a left-invariant vector field on a Lie group G, is a subgroup of G.

4. To a distribution Δ on M associate its "co-distribution" Δ^*

$$\Delta_p^* = \mathrm{Ann}(\Delta_p) = \{\phi_p \in T_p^*(M) : \ \phi_p(v) = 0 \ \ \forall v \in \Delta_p\}.$$

   Prove:

   a. $\dim \Delta_p^* = \dim M - \dim \Delta_p$ for all p.

   b. Δ is involutive iff for any local basis $\{\phi_i\}$ of Δ^* there exists 1-forms θ_i^j such that

$$d\phi_i = \sum_j \theta_i^j \wedge \phi_j.$$

5. Prove that a 1-form ϕ on M is exact if and only if for every closed curve c,

$$\int_c \phi = 0.$$

6. Let $U \subset \mathbb{R}^2$ be a domain and let $\chi : U \to R^3$, be an embedding endowed with the induced metric g. Let $E = g_{11}$, $F = g_{12}$, $G = g_{22}$ be the components of the induced metric relative to the coordinates u_1, u_2 on U.

   a. Show that the area element on $S = \chi(u) \subset \mathbb{R}^3$ is

   $$\Omega = \sqrt{EG - F^2}\; du_1 \wedge du_2$$

   b. Suppose that S is the graph of a function f, i.e.,

   $$\chi(u_1, u_2) = (u_1, u_2, f(u_1, u_2))$$

   Write $\int_U \Omega$ as an ordinary double integral on U, recovering the calculus formula

   $$\text{Area of}(S) = \int_U \sqrt{(\frac{\partial f}{\partial u_1})^2 + (\frac{\partial f}{\partial u_2})^2 + 1} \quad dudv$$

7. Assume there exists a bilinear map

   $$* : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

   so that for each $y \neq 0$, both of the maps

   $$x \longmapsto x * y$$
   $$x \longmapsto y * x$$

   are injective.

a. Show that

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$x \longmapsto x * e_1$$

is a diffeomorphism ($\{e_1 \dots e_n\}$ is the standard basis of $\mathbb{R}^n$. We assume that R^n has the standard Euclidean inner product).

b. Show that the projection of $\{x * e_2, \dots, x * e_n\}$ to the plane orthogonal to $x * e_1$ is linearly independent.

c. Show that the tangent bundle to S^{n-1} is trivial under the assumption that the pairing $*$ exists and is non-degenerate.