Advanced Calculus/Linear algebra basic exam

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Instructions: Do 7 of the 8 problems. Show your work. The passing standards are:

• Master’s level: 60% with three questions essentially, complete (including one question from each part);
• Ph.D. level: 75% with two questions from each part essentially complete.

Linear Algebra

1. Consider the matrix
   \[
   A = \begin{pmatrix}
   1 & 2 & -1 \\
   2 & 2 & c \\
   3 & 5 & -2
   \end{pmatrix}
   \]
   and \[b = \begin{pmatrix}
   -1 \\
   2 \\
   -1
   \end{pmatrix}.\]

   (a) Characterize the values \(c\) such that the matrix has a nontrivial nullspace (kernel).
   (b) For \(c = 1\), find all solutions \(x\) to the system \(Ax = b\).
   (c) For \(c = 0\), find all solutions \(x\) to the system \(Ax = b\).

2. Consider the linear transformation \(T : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) that reflects about the line \(y = x\).

   (a) What is the image \(T(1, 0)\) and \(T(0, 1)\).
   (b) Find the \(2 \times 2\) matrix \(A\) representation of \(T\) with respect to the basis \(\{(1, 0), (0, 1)\}\).
   (c) (i) Find the eigenvectors and their corresponding eigenvalues of \(T\) and (ii) diagonalize \(A\).
   (d) Find a \(2 \times 2\) matrix \(B\) that has no nonzero eigenvectors.

3. (a) Find an orthogonal basis for the span of the following three vectors \(S_2 = \{(1, 0, 1), (0, 1, 1), (1, 3, 3)\}\).
   (b) Write the vector \((1, 1, 2)\) in the orthogonal basis you found in (a).
   (c) Find a basis of the orthogonal complement \(W^\perp\) in \(\mathbb{C}^3\) of the vector space \(W = \text{span}((1, 1, 1))\).

4. Let \(V\) be a vector space and \(T : V \rightarrow V\) be a linear transformation.

   (a) Show that \(\{x \mid T(x) = x\}\) is a subspace of \(V\).
(b) Show that if $T^2 = T$, then for any $x \in V$, we have that $x - T(x)$ is in the nullspace (kernel) of $T$ denoted by ker($T$).

(c) Show that if $T^2 = T$ then $V = \{x \mid T(x) = x\} \oplus \ker(T)$.

(d) Characterize all linear transformations $T$ such that $T^2 = T$. 