UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Statistics
Wednesday, August 29, 2018

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level. Each answer is worth approximately the same number of points.

1. Let $X_1, \ldots, X_n$ be a sample of i.i.d. observations drawn from a distribution function $F(t) = P(X \leq t)$ for any fixed constant $t \in (-\infty, \infty)$. The empirical distribution function is defined as

$$\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq t),$$

where $I(\cdot)$ is the indicator function, i.e., $I(E) = 1$ if the event E occurs and 0 otherwise.

   (a) Find the exact distribution of $n\hat{F}_n(t)$ for any fixed t.

   (b) Find the mean and variance of $\hat{F}_n(t)$ for any fixed t.

   (c) Show that $\hat{F}_n(t)$ is an unbiased estimator of $F(t)$ for any fixed t.

   (d) Show that $\hat{F}_n(t)$ is a consistent estimator of $F(t)$ for any fixed t.

   (e) For any fixed t, show that

$$\sqrt{n}(\hat{F}_n(t) - F(t)) \xrightarrow{D} N(0, b(t))$$

   as $n \to \infty$, and specify the value of $b(t)$. Note that $N(a, b)$ represents the normal distribution with mean a and variance b, and $\xrightarrow{D}$ represents convergence in distribution.

2. Let $(x_1, Y_1), \ldots, (x_n, Y_n)$ be n pairs of independent samples. Consider the exponential regression model with mean θx_i, denoted as $Y_i \sim Exponential(\theta x_i)$,

$$f(y \mid \theta x) = \frac{1}{\theta x} e^{-\frac{y}{\theta x}}$$

where $\theta > 0$ is an unknown parameter, $i = 1, \ldots, n$ and $x_1, \ldots, x_n$ are positive known constants. Define $T_n = \sum_{i=1}^{n} \frac{Y_i}{x_i}$.

   (a) Write the likelihood for θ as a function of T_n, $x_1, \ldots, x_n$ and θ.

   (b) Is T_n is complete and sufficient for θ? why or why not?

   (c) Find the uniform minimum variance unbiased estimator (UMVUE) for θ and compute the variance of the UMVUE for θ.

(d) Suppose θ is distributed as Inverse Gamma $IG(\alpha, \beta)$ prior distribution,

$$f(\theta \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \frac{1}{\theta^{\alpha+1}} e^{-\frac{1}{\beta\theta}}$$

where $\theta > 0$, $\alpha > 1, \beta > 0$, the mean of θ is $\frac{1}{\beta(\alpha-1)}$, and $\Gamma(\cdot)$ denotes the gamma function.
Find the Bayes estimator of θ.

3. Let Y be a single observation with the probability density function

$$g(y \mid \theta) = (\theta/2)^{|y|}(1-\theta)^{1-|y|}$$

where $y = -1, 0, 1$ and $0 \leq \theta \leq 1$.

(a) Find a maximum likelihood estimator (MLE) for θ? Justify your answer.

(b) Show that $\mid Y \mid$ is an unbiased estimator of θ and compute the variance of $\mid Y \mid$.

(c) Under the assumption that $0 < \theta < 1$, compute the Crámer-Rao lower bound for $\mid Y \mid$ and check if it is equal to the variance of $\mid Y \mid$ computed in (b).

(d) Derive the form of the likelihood ratio test statistic for $H_0 : \theta = 0.2$ versus $H_1 : \theta \neq 0.2$.