DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - PROBABILITY
Fall 2018

Work all problems. Show all work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters level and 75 at the Ph.D. level. Points are shown for each question.

1) (20 points) You are given a bowl containing 14 chips of the same weight and size. Of the 14 chips, 3 are orange, 2 are blue, 5 are red and 4 are purple. The chips are all mixed and then one chip is selected at random from the bowl and its color is recorded. This chip is returned to the bowl and again all chips are mixed; after the mixing, a chip is randomly selected from the bowl and its color is again recorded. This chip is again returned to the bowl and the procedure is repeated in this manner. This procedure is named “sampling with replacement”. After the procedure is repeated *n* times, suppose that one examines the number of orange (X_1), blue (X_2), red (X_3) and purple (X_4) chips that are chosen.
a) What is the joint distribution of (X_1, X_2, X_3, X_4)?
b) What is the mean of X_4?
c) What is the variance of X_4?
d) Given that $n = 20$, what is the conditional distribution of (X_2, X_3, X_4) given that $X_1 = 3$?

2) (20 points) Questions related to the Central Limit Theorem.
a) State the Central Limit Theorem for a sequence of i.i.d. random variables.
b) Let $X \sim Binom(n, p)$ be a binomial random variable with *n* trials and probability of success *p*. Show by the Central Limit Theorem that for sufficiently large *n*, the sampling distribution of (X/n) is approximately normal with mean *p* and standard deviation $\sqrt{p(1-p)/n}$.

3) (25 points) Suppose that the random variable *X* has a Poisson distribution with mean μ. The probability mass function is:

$$f(x \mid \mu) = \frac{e^{-\mu}\mu^x}{x!}, \text{ for } \mu > 0;\ x = 0, 1, 2, \ldots.$$

You are given that

$$e^z = \sum_{k=0}^{\infty}\left(\frac{z^k}{k!}\right).$$

a) Find the moment-generating function for *X*.
b) If $X_1, X_2, \ldots, X_n$ are independent Poisson random variables with means $\mu_1, \mu_2, \ldots, \mu_n$, respectively, find the moment-generating function of $Y = \sum_{i=1}^{n} X_i$.
c) What is the distribution of *Y*?

4) (20 points) A random variable is exponentially distributed with parameter $\lambda > 0$ if its density function $f(x)$ is given by $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

a) Suppose X is an exponential random variable with parameter λ. For any t, $s > 0$, show that

$$P(X > t + s \mid X > s) = P(X > t).$$

b) For $i=1,2,\ldots,n$, let $X_i \sim \text{Exponential}(\lambda_i)$ be independent random variables. Show that

$$P(X_i < X_j \text{ for all } j \neq i) = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \ldots + \lambda_n}.$$

5) (15 points) Suppose that X and Y are discrete random variables defined on the same probability space satisfying $E(X) < \infty$ and $E(Y) < \infty$.

a) Show that

$$E_Y\left(E_{X/Y}\left(X \mid Y\right)\right) = E(X).$$

b) Suppose that $Y_1, Y_2, \ldots,$ are i.i.d. Bernoulli(p) random variables and that $N \sim \text{Poisson}(\lambda)$ is independent of $\left(Y_i\right)_{i=1}^{\infty}$. Find

$$E\left(\sum_{i=1}^{N} Y_i\right).$$