DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
AUGUST 2018

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correct.

PhD: 75% with at least three substantially correct.

1. Use Newton's method to find the root of the polynomial $p(x) = x^2 - 2x + 1$.

   (a) Does the Newton iteration converge for all initial guesses? Justify your answer.

   (b) Find the rate of convergence when the Newton iteraction converges.

2. (a) Derive a two-point integration formula to approximate

$$\int_{-1}^{1} f(x)(1+x^2)\mathrm{d}x$$

   that is exact when $f(x)$ is a polynomial of degree ≤ 3.

   (b) Compute the error in this approximation when $f(x) = x^4$.

3. Let $\{x_1, \cdots, x_n\}$ be n real and distinct interpolation nodes and let V be the Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}.$$

   (a) Let $l_i(x), i = 1, \cdots, n$ be the Lagrange interpolating polynomials. Use the fact $l_i(x_k) = \delta_{ik}$ to show that V is non-singular.

   (b) Let

$$\Phi_n(x) = \prod_{i=1}^{n}(x - x_i) = \sum_{j=1}^{n+1} a_j x^{j-1}.$$

   Outline an algorithm for finding the entries of V^{-1} based on finding the coefficients of $l_i(x), i = 1, \cdots, n$. (Hint: relate $l_i(x)$ to $q_i(x) := \Phi_n(x)/(x - x_i)$).

4. Let A be an $n \times n$ matrix. Define a symmetric matrix $\mathbf{T} = (\mathbf{A} + \mathbf{A}^T)/2$ and let $\mathbf{x} \in \mathbb{R}^n$ be any vector.

   (a) Show that $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{T}\mathbf{x}$.

   (b) Show that all eigenvalues of a symmetric positive definite matrix are strictly positive.

   (c) Show that $\mathbf{A}$ is positive definite if and only if $\mathbf{T}$ only has positive eigenvalues.

5. (a) Define the concept of *stability domain.*

   (b) Determine the stability domain for the leap-frog scheme $y_{n+1} - y_{n-1} = 2hf(t_n, y_n)$ for solving the ODE $y' = f(t, y)$, where h is the time step $h = t_{n+1} - t_n$.

(c) Determine the order of accuracy of the leap-frog scheme.

6. A matrix $\mathbf{A}$ is *strictly diagonally dominant* if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \quad \text{for } i = 1, \cdots, n\,.$$

Prove that if a matrix is strictly diagonally dominant, then no pivoting is necessary for Gaussian elimination. (Hint: Prove that the lower right corner of the partly processed matrix is also diagonally dominant.)

7. The Chebyshev polynomials satisfies $T_n(x) = cos(n \arccos x)$. Define the *inner product* of two continuous functions f and g by

$$(f, g) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) g(x) \mathrm{d}x\,.$$

(a) Show that $T_n(x), n \geq 0$ are orthogonal polynomials.

(b) Given $h(x) \in C([-1, 1])$. Solve the minimization problem

$$\min_{p(x)\in\mathbb{P}^n(x)} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} (h(x) - p(x))^2 \mathrm{d}x\,,$$

where $\mathbb{P}^n(x)$ is the space of all polynomials with degree $\leq n$.