DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC QUALIFYING EXAM – APPLIED MATHEMATICS

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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially complete solutions.

1) [20 points] (a) Find a harmonic function $u(r, \theta)$ inside a wedge defined by the three sides: $\theta = 0$, $\theta = \beta$ and $r = a$, which satisfies the boundary conditions

$$u(r, \theta = 0) = 0, \quad u(r, \theta = \beta) = 0, \quad u(r = a, \theta) = g(\theta).$$

(b) Apply this general solution to the special case of a semi-disk with $\beta = \pi/4$, $a = 1$ and $g(\theta) = 15\sin(4\theta) - 2\sin(16\theta)$, to find $u(r, \theta)$ in this case.

2) [20 points] (a) For each of the following equations, state the *order* and whether is *linear* or *nonlinear*. Explain/justify your answer.

- $u_t - 2u_{xx} = x^2$
- $u_t - 2u_{xx} + xu = 0$
- $u_t - 2u_{xx} + u^4 = 0$
- $u_t - u_{xxt} + uu_x = 0$
- $iu_t - u_{xx} + \frac{u}{3x} = 0$
- $u_y + 2e^{-y}u_x = 0$
- $u_t - u_{xxxx} + \sqrt{1 + u^2} = 0$

(b) Consider the PDE

$$u_t + u^r u_x = 0$$

where $r > 0$ is an integer constant. The initial condition is given $u(x,0) = \phi(x)$. Knowing only $\phi(x)$ (and that it possesses a spatially decreasing part), find an expression for the *minimal time* at which the solution will form a shock.

3) [20 points] Investigate the following "rabbit vs. sheep" problem:

$$\dot{x} = x(3 - 2x - 2y),$$
$$\dot{y} = y(2 - x - y)$$

(a) Identify all the fixed points and investigate their stability.

(b) Draw the nullclines and sketch the phase portrait.

(c) Indicate the basins of attraction of any stable fixed points.

4) [20 points] Consider the dynamical system as a function of its parameter μ:

$$\dot{x} = \mu x + y + \sin(x)$$
$$\dot{y} = x - y$$

1. Illustrate that the system has a bifurcation at $\mu = -2$ and identify its nature.
2. Does the system have any additional bifurcations ? Can you sketch a full bifurcation diagram of x as a function of μ ?

5) [20 points] Let $u_1(x,t)$ and $u_2(x,t)$ be solutions to the heat equation

$$u_t = k\, u_{xx}\,, \quad \text{in} \quad R = [0, L] \times [0, \infty)\,,$$

with initial and boundary conditions:

$$u_1(x,0) = f_1(x),\; u_1(0,t) = g_1(x),\; u_1(L,t) = h_1(t)\,,$$

and

$$u_2(x,0) = f_2(x),\; u_2(0,t) = g_2(x),\; u_2(L,t) = h_2(t)\,,$$

respectively.

Assume that $f_1 \leq f_2$, $g_1 \leq g_2$ and $h_1 \leq h_2$. Prove that then $u_1 \leq u_2$ in the region $R = [0, L] \times [0, \infty)$.

6) [20 points] (a) Let $g \in C^2([0,1])$ be a given function and consider the equation:

$$u_t - u_{xx} = 0, \qquad 0 < x < 1, \;\; t > 0 \tag{1}$$

with initial and boundary conditions

$$u(x,0) = g(x), \quad 0 < x < 1\,,$$

$$u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0\,,$$

respectively. Use the method of separation of variables to find the general solution $u = u(x,t)$ as a Fourier series. Give the form of the coefficients A_n, $n \geq 1$ in the series in terms of g.

(b) What is the $\lim_{t\to\infty} u(x,t)$ equal to? Justify your answer (you may formally take the limit inside the series).

7) [20 points] (a) Find real numbers a, b so that that $V(x,y) = ax^2 + by^2$ is a strict Liapunov function for the origin of the system

$$\dot{x} = y - x^3, \;\; \dot{y} = -x - y^3$$

(b) Explain intuitively why part (a) implies that the system has no (non-constant) periodic orbits.