Department of Mathematics and Statistics
University of Massachusetts

Topology qualifying exam

Thursday, August 30, 2018

Answer all seven questions. Justify your answers.
Passing standard: 70% with four questions essentially complete.

1. Suppose $f, g : X \to Y$ are continuous maps and Y is Hausdorff. Show that the set $\{x \in X \mid f(x) = g(x)\}$ is closed in X. Give a counter-example if Y is not Hausdorff.

2. Suppose that A, B are closed subsets of a topological space X, and that both $A \cup B$ and $A \cap B$ are connected. Show that A and B are connected.

3. Given subsets $A, B \subset \mathbb{R}^n$ define their sum to be
$$A + B = \{a + b \mid a \in A,\ b \in B\}.$$
   (a) Show that if A and B are compact, then $A + B$ is compact.
   (b) Show that if A is compact and B is closed, then $A + B$ is closed.

4. Suppose that M is a connected manifold of dimension at least 3, and $p \in M$. Show that the inclusion $M \setminus \{p\} \hookrightarrow M$ induces an isomorphism $\pi_1(M \setminus \{p\}) \cong \pi_1(M)$.

5. Show that any map $\mathbb{RP}^2 \to S^1 \times S^1$ is nullhomotopic. Find, with proof, a map $S^1 \times S^1 \to \mathbb{RP}^2$ which is not nullhomotopic.

6. Let W be the space obtained by attaching two 2-cells to S^1, one by map $z \mapsto z^4$ and the other by the map $z \mapsto z^7$.
   (a) Compute the homology groups of W with $\mathbb{Z}$ coefficients.
   (b) Is W homotopy equivalent to S^2? Justify your answer.

7. Show that $S^2 \times S^2$ and $S^2 \vee S^2 \vee S^4$ have isomorphic cohomology groups in all degrees and with any coefficients, but are not homotopy equivalent.