Department of Mathematics and Statistics
University of Massachusetts
Topology qualifying exam
Thursday, August 30, 2018

Answer all seven questions. Justify your answers.
Passing standard: 70% with four questions essentially complete.

1. Suppose $f, g : X \to Y$ are continuous maps and $Y$ is Hausdorff. Show that the set $\{x \in X \mid f(x) = g(x)\}$ is closed in $X$. Give a counter-example if $Y$ is not Hausdorff.

2. Suppose that $A, B$ are closed subsets of a topological space $X$, and that both $A \cup B$ and $A \cap B$ are connected. Show that $A$ and $B$ are connected.

3. Given subsets $A, B \subset \mathbb{R}^n$ define their sum to be
   
   $$A + B = \{a + b \mid a \in A, b \in B\}.$$

   (a) Show that if $A$ and $B$ are compact, then $A + B$ is compact.
   (b) Show that if $A$ is compact and $B$ is closed, then $A + B$ is closed.

4. Suppose that $M$ is a connected manifold of dimension at least 3, and $p \in M$. Show that the inclusion $M \setminus \{p\} \hookrightarrow M$ induces an isomorphism $\pi_1(M \setminus \{p\}) \cong \pi_1(M)$.

5. Show that any map $\mathbb{R}P^2 \to S^1 \times S^1$ is nullhomotopic. Find, with proof, a map $S^1 \times S^1 \to \mathbb{R}P^2$ which is not nullhomotopic.

6. Let $W$ be the space obtained by attaching two 2-cells to $S^1$, one by map $z \mapsto z^4$ and the other by the map $z \mapsto z^7$.

   (a) Compute the homology groups of $W$ with $\mathbb{Z}$ coefficients.
   (b) Is $W$ homotopy equivalent to $S^2$? Justify your answer.

7. Show that $S^2 \times S^2$ and $S^2 \vee S^2 \vee S^4$ have isomorphic cohomology groups in all degrees and with any coefficients, but are not homotopy equivalent.