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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all the results that you use in your proofs and <u>verify</u> that these results apply.

5. Show all your work and <u>justify</u> the steps in your proofs.

6. Please write your full work and answers <u>clearly</u> in the blank space under each question and on the blank page after each question.

Conventions

1. If a measure is not specified, use Lebesgue measure on $\mathbb{R}^d$. This measure is denoted by m or $m_{\mathbb{R}^d}$.

2. If a σ-algebra on $\mathbb{R}^d$ is not specified, use the Borel σ-algebra.

1. Let $\{E_n\}_{n\geq 1}$ be a countable collection of measurable sets in $\mathbb{R}^d$. Define

$$\limsup_{n\to\infty} E_n \ := \ \{x \in \mathbb{R}^d \ : \ x \in E_n, \ \text{for infinitely many} \ n\}.$$

$$\liminf_{n\to\infty} E_n \ := \ \{x \in \mathbb{R}^d \ : \ x \in E_n, \ \text{for all but finitely many} \ n\}.$$

a) Show that

$$\limsup_{n\to\infty} E_n \ = \ \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \qquad \liminf_{n\to\infty} E_n \ = \ \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} E_j.$$

b) Show that

$$m(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} m(E_n),$$

and that

$$m(\limsup_{n\to\infty} E_n) \geq \liminf_{n\to\infty} m(E_n) \quad \text{provided that } m(\bigcup_{n=1}^{\infty} E_n) < \infty.$$

2. Let E be a measurable subset of $\mathbb{R}$ with $m(E) > 0$. Prove that for each $0 < \alpha < 1$ there exists an interval I in $\mathbb{R}$ so that

$$m(E \cap I) \geq \alpha m(I).$$

<u>Hints</u> Express $m(E)$ as an infimum over all $m(\mathcal{O})$ with $\mathcal{O} \supseteq E$, $\mathcal{O}$ open. Recall that an open set $\mathcal{O}$ can be written as the countable union of disjoint open intervals.

3. Consider the function $f(x, y) := e^{-xy} - 2e^{-2xy}$ where $x \in (1, \infty)$ and $y \in (0, 1)$.

   (a) Prove that for a.e. $y \in (0, 1)$ $\;f^y$ (defined as $f^y(x) = f(x, y)$) is integrable on $(1, \infty)$ with respect to $m_{\mathbb{R}}$.

   (b) Prove that for a.e. $x \in (1, \infty)$ $\;f^x$ (defined as $f^x(y) = f(x, y)$) is integrable on $(0, 1)$ with respect to $m_{\mathbb{R}}$.

   (c) Prove that $f(x, y)$ is not integrable on $(1, \infty) \times (0, 1)$ with respect to $m_{\mathbb{R}^2}$.

   <u>Hint for c).</u> You can use Fubini.

4. (a) Let $\{e_n\}_{n=1}^{N}$ be an orthonormal collection of functions in $L^2([a,b])$. Given $f \in L^2([a,b])$ find the values of $a_k \in \mathbb{R}$ which minimize $\|f - \sum_{n=1}^{N} a_n e_n\|_{L^2([a,b])}$.

   (b) Suppose $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2([a,b])$. Show that if $\{\varphi_n\}_{n=1}^{\infty}$ is another collection of functions (not necessarily orthonormal) in $L^2([a,b])$ such that

$$\sum_{n=1}^{\infty} \|e_n - \varphi_n\|_{L^2([a,b])}^2 < 1$$

   then $\{\varphi_n\}_{n=1}^{\infty}$ is also a complete system: that is, show that if $f \in L^2([a,b])$ is orthogonal to φ_n for every $n \geq 1$ then f is the zero function.

5. Let $f : [a, b] \to \mathbb{R}$ be a given function.

   (a) Show that if f is absolutely continuous then it has finite total variation.

   (b) Show that if f is Lipschitz then f has finite total variation. Recall that f is said to be Lipschitz if there exists a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all x, y in $[a, b]$.

   (c) Is it possible for f to be continuous but not have finite total variation? Justify your answer.

6. (a) Consider $f_n(x) := \chi_{[n,n+1]}(x)$ be a sequence in $L^1(\mathbb{R})$. Show that $\|f_n\|_{L^1} = 1$ for all $n \geq 1$ and that $f_n \to 0$ pointwise but $f_n \not\to 0$ weakly in L^1.

(b) For every $n \geq 1$ let $f_n(x) := \cos(2\pi nx)$ be a sequence in $L^2([0,1))$. Show that $f_n \to 0$ weakly in L^2 but $f_n \not\to 0$ a.e. x.

<u>Hints.</u> For the first part of (b) recall that $\cos(2\pi nx) = \frac{(e^{2\pi inx}+e^{-2\pi inx})}{2}$. For the second part, you may argue by contradiction or if arguing directly you may use that the sequence $\{\overline{nx} : n \geq 1\}$ is dense in $[0,1)$ for x irrational (here we denoted by $\overline{nx} := nx(\text{mod}1)$).

(c) Let $f_n(x) := n\chi_{(0,\frac{1}{n})}(x)$ be a sequence in $L^2([0,1])$. Show that $f_n \to 0$ a.e. x and in measure but $f_n \not\to 0$ weakly in L^2.

7. Let $\mathcal{H}$ be a Hilbert space, and $L : \mathcal{H} \to \mathcal{H}$ a linear function.

   (a) Show that L is bounded if and only if it is continuous.

   (b) Suppose $\|L\| < 1$, where $\|\cdot\|$ denotes the operator norm, and let $I : \mathcal{H} \to \mathcal{H}$ be the identity operator (that is, $I(h) = h$ for every $h \in \mathcal{H}$). Show that $I - L$ is invertible.

   Hint: Think in terms of power series.

8. Let $1 \leq p < \infty$ fixed. For $f \in L^p(\mathbb{R}^d)$ consider the distribution function $\lambda_f : [0, \infty] \to [0, \infty]$ defined by

$$\lambda_f(a) := m(\{ x \in \mathbb{R}^d \ : \ |f(x)| > a \}).$$

Recall that λ_f is decreasing and right continuous and that $\int_{\mathbb{R}^d} |f(x)|^p \, dx = \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha$.

(a) Show that if $\sum_{k=-\infty}^{\infty} 2^{kp} \lambda_f(2^k) < \infty$ then $f \in L^p(\mathbb{R}^d)$.

Hint: Consider the sets $E_j := \{ x \in \mathbb{R}^d \ : \ 2^j < |f(x)| \leq 2^{j+1} \}$

(b) Show that if $f \in L^p(\mathbb{R}^d)$ then $\sum_{k=-\infty}^{\infty} 2^{kp} \lambda_f(2^k) < \infty$.

Hint: One approach is to note that $1 = \int_0^\infty 2^{1-n} \chi_{[2^{n-1}, 2^n)}(\alpha) \, d\alpha$ and use it to rewrite

$$\sum_{k=-\infty}^{\infty} 2^{kp} \lambda_f(2^k) = \int_0^\infty 2^{kp} 2^{1-k} \lambda_f(2^k) \, \chi_{[2^{k-1}, 2^k)}(\alpha) \, d\alpha.$$