DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ADVANCED EXAM — ALGEBRA

AUGUST 2018

Passing Standard: To pass the exam it is sufficient to solve five problems including a least one problem from each of the three parts. Show all your work and justify your answers carefully.

1. Group theory

1. Let p, q be odd primes. Prove that a group of order $2pq$ is solvable.
Note: p, q may or may not be distinct!

2. Let p be a prime. Determine the number of conjugacy classes of a **non-Abelian** group G of order p^3.

3. Let G be a group of order 60. Assume that the center $Z(G)$ has order divisible by 4. Show that G is abelian.

2. Commutative algebra

4. Let R with a commutative ring with 1.
(a) Suppose M is a finitely generated free R-module. Show that $\mathrm{Hom}_R(M, R)$ is a finitely generated free R-module.
(b) Suppose M is a free R-module, but **not** finitely generated. Prove or give a counter-example to the following statement: $\mathrm{Hom}_R(M, R)$ is a free R-module.
(c) Suppose M is a finitely generated R-module, but **not** free. Prove or give a counter-example to the following statement: $\mathrm{Hom}_R(M, R)$ is a finitely generated R-module.

5. Let R be a principal ideal domain and let A, B, C be finitely generated R-modules. Show that if $A \oplus B \cong A \oplus C$, then $B \cong C$.

6. Consider the ring

$$R = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a \equiv b \pmod 5\}.$$

(1) Show that the homomorphism

$$f : \mathbf{Z}[x] \to R$$

sending 1 to $(1, 1)$ and x to $(5, 0)$ is surjective with kernel $(x^2 - 5x)$.
(2) Determine all prime ideals of R containing $f(3) = 3 \cdot 1_R$.
(3) Determine all prime ideals of R containing $f(5) = 5 \cdot 1_R$.

3. Field theory and Galois theory

7. Let L/K be a finite extension of fields such that $L = K(\alpha, \beta)$ for some elements $\alpha, \beta \in L$. Suppose $[K(\alpha) : K]$ and $[K(\beta) : K]$ are relatively prime.
(a) Show that the minimal polynomial of α over K is irreducible over $K(\beta)$.
(b) Show that $[L : K] = [K(\alpha) : K][K(\beta) : K]$.

8. Show that $\mathbf{Q}(\sqrt{5} + \sqrt{11})$ is Galois over $\mathbf{Q}$ and determine its Galois group.
Hint: Obviously $\mathbf{Q}(\sqrt{5} + \sqrt{11})$ is a subfield of $\mathbf{Q}(\sqrt{5}, \sqrt{11})$. What does that say about $[\mathbf{Q}(\sqrt{5} + \sqrt{11}) : \mathbf{Q}]$?

9. Let p be a prime and let K be a finite field of order p^{30}. Determine the number of elements $\alpha \in K$ such that $K = \mathbf{F}_p(\alpha)$.