DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST

COMPLEX ANALYSIS EXAM

AUGUST 2017

- Each problem is worth 10 points.
- Do 8 of the following 10 problems.
- Passing Standard:
  - Master's Level: 45/80 with three questions essentially complete
  - Ph.D. Level: 55/80 with four questions essentially complete

(1) Find the Laurent expansion of

$$f(z) = \frac{4z-2}{z(z-1)}$$

about $z = 0$ in $\{0 < |z| < 1\}$.

(2) (a) Let $U \subseteq \mathbf{C}$ be open and fix $a \in U$. Let $f : U - \{a\} \to \mathbf{C}$ be holomorphic. Define the residue of f at a.

(b) Determine the type of each singularity (removable, pole, or essential), and calculate the residues of the following functions.

(i) $\frac{\sin z}{z(z-1)^2}$

(ii) $\frac{1}{e^z-1} + \frac{z}{(z-1)(z-i)}$

(3) Compute the integral

$$\int_C \frac{dz}{(z-3)(2+3z)^3(i-2z)^2} dz$$

where C is the circle $\{|z| = 1\}$ oriented counter-clockwise.

(4) Find the number of roots of the polynomial

$$P(z) = 2z^5 - z^3 + z + 7$$

in the annulus $\{1 < |z| < 2\}$.

(5) Suppose $f(z) = \sum_{n=0} a_n z^n$ has radius of convergence $R > 0$. Show that

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

defines an entire function and that for each fixed $0 < r < R$, there is a constant M such that $|g(z)| \leq M e^{|z|/r}$ for all $z \in \mathbf{C}$.

(6) Set $U := \{z \ : \ |z| < 1 \text{ and } \mathrm{Im}(z) > 0\}$ to be the upper-half of the unit disk. Find a linear fractional transformation f, mapping U onto itself, and satisfying $f(i) = 0$ and $f(1) = -1$. Prove that f is unique.

(7) Evaluate the integral

$$\int_0^{+\infty} \frac{\ln(x)}{x^2+1}\, dx.$$

Hint: Consider a contour consisting of two semi-circles, centered at the origin, and two line-segments along the x-axis. Include a proof, that the contour you chose can be used to evaluate the integral.

(8) Prove that the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

defines a meromorphic function on $\mathbf{C}$, periodic of period 1 and with double poles at the integers and no other poles.

(9) Let $f : \mathbf{C} \to \mathbf{C}$ be a meromorphic function that has periods 1 and τ with $\mathrm{Im}(\tau) > 0$. Thus $f(z + j + k\tau) = f(z)$ for all $z \in \mathbf{C}$ and all $j, k \in \mathbf{Z}$.
 (a) Prove that if f has no singularities whatsoever, then f must be constant.
 (b) Assume that f has no poles on the boundary C of the set

$$S = \{s + t\tau \mid 0 \le s < 1, 0 \le t < 1\}.$$

 Prove that $\int_C \frac{f'(z)}{f(z)} dz = 0$.

(10) Show that the function $f(z) = z - 3 + 2e^{-z}$ has precisely one zero in the right half plane $\mathrm{Re}(z) > 0$.