DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
MASTERS OPTION EXAM APPLIED MATH

Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct

1. Consider the dynamical system

$$\frac{dX}{dt} = (X-1)(X^2 - r)$$

where $r \in \mathbb{R}$ is a parameter.

   (a) Find and classify the bifurcation points.

   (b) Find and plot the equilibrium solution(s) against the parameter r (use solid lines for stable curves and dashed lines for unstable curves).

2. Solve the following Poisson equation

$$-u_{xx} - u_{yy} = \sin(x)\sin(2y) \qquad \text{in } (0,\pi)\times(0,\pi)$$
$$u(0,y) = u(\pi,y) = u(x,0) = 0, \qquad u(x,\pi) = \sin(y)$$

3. (a) Consider the equation

$$y''' + y' = 0$$

   with initial conditions $y(0) = 1, y'(0) = 0, y''(0) = 0$. Note that $y(t) = 1$ solves this equation. Consider a perturbation $f(t) = \epsilon \sin(\omega t)$ applied to the right-hand side of the equation, where $\epsilon \ll 1$ and $\omega > 0$. What effect does this perturbation have on the qualitative behavior of the solution?

   (b) Consider now the equation

$$y''' - y' = 0,$$

   with the same initial conditions as in the previous part. How does perturbation $f(t) = \epsilon \sin(\omega t)$ affect the solution?

4. We consider the hyperbolic conservation law

$$u_t + ((1-u)u)_x = 0.$$

This can model, for example, traffic density u, where the cars have speed $1-u$, moving slower in heavy traffic. We consider the initial condition

$$u(x,0) = \begin{cases} 1/4 & x \leq 1/4 \\ x & 1/4 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}.$$

Solve the equation using the method of characteristics. A shock will form. Resolve the motion of the shock.

5. (a) Find $u(x,t)$ that satisfies:

$$u_t = ku_{xx}$$

and also $u(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = x$.

(b) Apply Parseval's identity for the Fourier series decomposition of $f(x) = x$ (that you computed in part a) to determine

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

6. Two species of fish that compete with each other for food, but do not prey on each other are the bluegill and redear, Suppose that the pond is stocked with bluegill and redear and let x and y be the populations of bluegill and redear respectively, at time t. Suppose that the competition is modeled by the ODEs

$$\frac{dx}{dt} = x(a_1 - b_1 x - c_1 y)$$

$$\frac{dy}{dt} = y(a_2 - b_2 y - c_2 x)$$

If $a_2/c_2 > a_1/b_1$ and $a_2/b_2 > a_1/c_1$, show that the only possible equilibrium populations in the pond have either (1) no fish or (2) no bluegill or (3) no redear. What happens as time approaches infinity?

7. Consider the circuit equation

$$LI'' + RI' + IC = 0$$

where $L, C > 0$ and $R \geq 0$.

(a) Rewrite the equation as a two-dimensional system.

(b) Show that the origin is asymptotically stable if $R > 0$ and neutrally stable if $R = 0$.

(c) Classify the fixed point at the origin, depending on whether $R^2C - 4L$ is positive, negative, or zero, and sketch the phase portrait in all three cases.