DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
ADVANCED EXAM - DIFFERENTIAL EQUATIONS

Friday, September 1st, 2017
10:00AM – 1:00PM
LGRT 1634

Do **five** of the following seven problems. All problems carry equal weight.
Passing level: 75% with at least three complete solutions.

(1) Consider the nonlinear autonomous system given by

$$\dot{x} = y,$$
$$\dot{y} = -\sinh(x).$$

(a) Discuss the linear stability of the equilibrium point at $(0,0)$.
(b) Using a Lyapunov function, show that $(0,0)$ is also nonlinearly stable.
(c) Is $(0,0)$ asymptotically stable? Explain why or why not.

(2) The Brusselator system is given by

$$\dot{x} = -4x + 1 + x^2 y, \ \dot{y} = 3x - x^2 y.$$

(a) Determine the fixed points, as well as their type.
(b) Consider the trapezoid determined by the vertices

$$(\tfrac{1}{4}, 0), \ (13, 0), \ (1, 12), \ (\tfrac{1}{4}, 12)$$

and show that it is a positively invariant region.
(c) Show the system has a nontrivial periodic solution.

(3) Suppose you know that a two dimensional system $\dot{x} = Ax$ has a general solution of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} 2 + c_2 e^{-t} 5 \\ c_1 e^{-t} 3 + c_2 e^{-t} 5 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

Using this information, determine the matrix A, and the exponential e^{tA}.

(4) Provide a general formula for the solution of the Cauchy problem

$$u_t + u_x = f(x,t), \quad x \in \mathbb{R}, \ t > 0,$$
$$u = 0, \quad x \in \mathbb{R}, \ t = 0.$$

and provide the explicit solution in the case $f = e^{-t} \sin(x)$.

(5) Let $u : \Omega \subset \mathbb{R}^2 \mapsto \mathbb{R}$ be a harmonic function. Define
$$\Omega_\delta := \{x \in \Omega \mid d(x, \partial\Omega) > \delta\},$$
and, in Ω_δ, define the function
$$u_\delta(x) = \int_{\mathbb{R}^2} K(|y|/\delta) u(x-y)\, dy,$$
where $K : \mathbb{R} \mapsto \mathbb{R}$ is an even, positive, C^∞ function with compact support in $(-1,1)$, such that $K \equiv 1$ in neighborhood of 0 and
$$\int_{\mathbb{R}^2} K(|x|)\, dx = 1.$$

(a) Show that $u(x) = u_\delta(x) \ \ \forall\, x \in \Omega_\delta$. *Hint: Write integral in polar coordinates.*
(b) Let $x \in \Omega_r$ for some $r > 0$, then show that
$$|\nabla u(x)| \leq \frac{C_1}{r} \sup_{B_r(x)} |u|, \quad |D^2 u(x)| \leq \frac{C_2}{r^2} \sup_{B_r(x)} |u|.$$
where C_1 is determined by $\|K'\|_{L^\infty}$ and C_2 is determined by $\|K''\|_{L^\infty}$.

(6) Consider the problem with Robin boundary conditions
$$-\Delta u + \gamma u = f \text{ in } D,$$
$$\frac{\partial u}{\partial \mathbf{n}} + \alpha u = 0 \text{ on } \partial D.$$
Here, $D \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary, $\mathbf{n}$ is its outer normal vector, and $f \in L^2(D)$. Then,

(a) Write down the Sobolev space and weak formulation corresponding to this problem
(b) Using the right test function, prove the energy identity
$$\int_D |\nabla u|^2\, dx + \alpha \int_{\partial D} u^2\, d\sigma + \gamma \int_D u^2\, dx = \int_D f u\, dx.$$
(c) Assuming that both α and γ are strictly positive, show there exists one, and only one, weak solution.

(7) Given a function $u : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$, define its kinetic and potential energy by
$$K(t) := \int_{\mathbb{R}} (\partial_t u(t,x))^2\, dx, \quad P(t) := \int_{\mathbb{R}} (\partial_x u(t,x))^2\, dx.$$
Now, suppose that u solves the one dimensional wave equation $\partial_{tt} u = \partial_{xx} u$ and that it's initial data $u(0,x)$ and $u_t(0,x)$ is smooth with compact support. Show there is a $T > 0$ such that
$$K(t) = P(t), \quad t > T.$$