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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all the results that you use in your proofs and <u>verify</u> that these results apply.

5. Show all your work and <u>justify</u> the steps in your proofs.

6. Please write your full work and answers <u>clearly</u> in the blank space under each question and on the blank page after each question.

Conventions

1. If a measure is not specified, use Lebesgue measure on $\mathbb{R}^d$. This measure is denoted by m.

2. If a σ-algebra on $\mathbb{R}^d$ is not specified, use the Borel σ-algebra.

1. (a) Let $\mathcal{A}$ be a σ-algebra on a set X and let $\{B_k\}_{k=1}^{\infty}$ be a sequence of pairwise disjoint sets. For each $n \in \mathbb{N}$ define $A_n := \bigcup_{k=n}^{\infty} B_k$. Prove that $\bigcap_{n=1}^{\infty} A_n = \emptyset$

(b) Let $\mathcal{A}$ be a σ-algebra on a set X and assume that $\mu : \mathcal{A} \to [0, \infty]$ has the following properties:

(i) If $A_1, A_2 \in \mathcal{A}$ with $A_1 \cap A_2 = \emptyset$, then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.

(ii) If $\{A_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{A}$ such that $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} A_n = \emptyset$, then $\lim_{n\to\infty} \mu(A_n) = 0$

Prove that μ is a positive measure on X.

Hint for (b). To show countable additivity, use (a) and note that $\bigcup_{k=1}^{\infty} B_k$ is the disjoint union of $\bigcup_{k=1}^{n-1} B_k$ and A_n provided $n \geq 2$.

2. (a) Compute the following limit justifying all steps,

$$\lim_{n\to\infty}\int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\,dx.$$

(b) Suppose $g:\mathbb{R}\to[0,\infty)$ is in $L^1(\mathbb{R})$ and $f:\mathbb{R}\to\mathbb{R}$ is continuous and bounded. Show that

$$\lim_{n\to\infty}\int_{\mathbb{R}} n\,g(nx)\,f(x)\,dx \;=\; f(0)\,\|g\|_{L^1(\mathbb{R})}.$$

3. (a) Suppose that $\{f_n\}_{n\geq 1}$, $\{g_n\}_{n\geq 1}$, f, g are functions in $L^1(\mathbb{R}^d)$ and that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for a.e. x in $\mathbb{R}^d$ as $n \to \infty$. Show that if $|f_n| \leq g_n$ for all $n \geq 1$ and $\lim_{n\to\infty} \int g_n \, dx = \int g \, dx$ then $\lim_{n\to\infty} \int f_n \, dx = \int f \, dx$.

   Hint for (a). Apply Fatou's lemma to $g_n \pm f_n$.

   (b) Suppose $\{f_n\}_{n\geq 1}$, f are functions in $L^1(\mathbb{R}^d)$ and that $f_n(x) \to f(x)$ for a.e. x in $\mathbb{R}^d$. Prove that $f_n \to f$ in $L^1(\mathbb{R}^d)$ if and only if $\|f_n\|_{L^1(\mathbb{R}^d)} \to \|f\|_{L^1(\mathbb{R}^d)}$.

   Hint for (b). Use part (a).

4. (a) Let (X, μ) be a measure space and let $f \in L^1(X, d\mu)$. Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|\int_A f \, d\mu| < \varepsilon,$$

whenever A is a measurable subset of X with $\mu(A) < \delta$.

(b) Consider $\mathbb{R}$ together with the Lebesgue measure m. Suppose that λ is a finite positive measure on $\mathbb{R}$ which is absolutely continuous with respect to m; that is $\lambda \ll m$. Prove that the function $F(x) := \lambda((-\infty, x]),\ x \in \mathbb{R}$ is *uniformly* continuous.

Hint for (b). Use part (a).

5. Let N be some fixed positive integer and for any $\sigma > 0$ consider the function:

$$\psi_\sigma(s) := \begin{cases} (s/\sigma)^N & \text{if} \quad 0 < s \leq \sigma, \\ 0 & \text{if} \quad \sigma < s < \infty \end{cases}$$

(a) Show that for any $a > 0$, $\psi_{a\sigma}(s) = \psi_\sigma(a^{-1}s)$.

(b) Suppose that g is a non-negative function in $L^1([0,\infty), \frac{dx}{x})$. That is, $g \geq 0$ is integrable on $[0,\infty)$ with respect to the measure $\frac{dx}{x}$.

Show that,

$$\int_0^\infty \int_s^{2s} \psi_\sigma(s)\, g(t)\, \frac{dt}{t}\, \frac{ds}{s} = \int_\sigma^{2\sigma} \int_0^\infty \psi_u(s) g(s) \frac{ds}{s} \frac{du}{u}.$$

Hint for (b). Use a change of variables, Fubini-Tonelli (justify), and part (a).

6. Let $\mathcal{H}$ be a real Hilbert space $\mathcal{H}$ with inner product $\langle \cdot\,, \cdot \rangle$ and let S be a subset of $\mathcal{H}$.

We denote by $\overline{S}$ the smallest closed subspace of $\mathcal{H}$ containing S. We define the set

$$S^{\perp} := \{u \in \mathcal{H} : \langle u\,, v \rangle = 0 \text{ for all } v \in S\}$$

(a) Prove that $S^{\perp}$ is a closed subspace of $\mathcal{H}$

(b) Prove that $S \subset (S^{\perp})^{\perp}$ and that $\overline{S}^{\perp} \subset S^{\perp}$

(c) Use the fact for any closed subspace M of $\mathcal{H}$, $(M^{\perp})^{\perp} = M$ (which you need not prove) together with (b) to prove that $(S^{\perp})^{\perp} = \overline{S}$.

7. (a) Let f be an integrable function defined on $[a, b]$ and let ϕ be a continuous convex function defined on $\mathbb{R}$. Prove that

$$\phi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \le \frac{1}{b-a}\int_a^b \phi(f(x))\,dx.$$

Hint for (a). Note that if ϕ is convex on $\mathbb{R}$, then for every $(x_0, \phi(x_0))$ on the graph of ϕ, there exists an $\alpha = \alpha(x_0)$ in $\mathbb{R}$ such that $\phi(x) \ge \alpha(x - x_0) + \phi(x_0)$ for all $x \in \mathbb{R}$. You need **not** prove this. To prove the desired inequality, pick a suitable x_0.

(b) Show that if $f \in L^q([0,1])$, $q > 0$, then

$$\int_0^1 \log|f|\,dx \le \log\|f\|_{L^q([0,1])}.$$

Hint for (b). Use part (a) with $\phi(t) = e^t$ and an appropriate integrable function.

8. (a) Let $1 < p < q < \infty$. Show that $L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ with the norm $\|f\|_{L^p \cap L^q} = \|f\|_{L^p} + \|f\|_{L^q}$ is a Banach space.

(b) Let $1 < p < r < q < \infty$. Show that $L^p(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \subseteq L^r(\mathbb{R}^d)$ and that the inclusion map is continuous with respect to the norm in the previous part.