Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

Wednesday August 31, 2016

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) A family of subsets $\mathcal{F}$ of a space X is said to be *locally finite* if any point $x \in X$ has an open neighborhood U such that only finitely many $S \in \mathcal{F}$ have nonempty intersection with U.

(a) Show that if $\mathcal{F}$ is a locally finite family of closed sets then the set

$$\bigcup_{C \in \mathcal{F}} C$$

is closed.

(b) Show that if $\mathcal{F}$ is a locally finite closed cover of X and $f\colon X \to Y$ is a function with the property that for each $C \in \mathcal{F}$, the restriction $f|_C$ is continuous, then f is continuous.

(2) (a) Give the definitions of: "locally connected" and "connected component".

(b) Let X be locally connected and compact. Show that X has only finitely many connected components.

(3) Let C be a subset of a topological space X.

(a) Prove that if C is connected, then the closure $\overline{C}$ is connected.

(b) Prove or disprove: if C is connected, then the interior of C is connected.

(4) A map $f\colon X \to Y$ is called *proper* if for any compact set $C \subset Y$, the inverse image $f^{-1}(C)$ is compact.

(a) Show that if f is surjective and closed and the fiber $f^{-1}(y)$ is compact for every $y \in Y$, then f is proper.

(b) Let $f\colon X \to Y$ be a proper map between metric spaces. Show that a sequence $\{x_n\}$ in X has a convergent subsequence if and only if $\{f(x_n)\}$ has a convergent subsequence.

(5) Let X be a path-connected subset of $\mathbb{R}^2 \setminus \{0\}$ which contains the unit circle S^1. Prove that $\pi_1(X)$ contains a subgroup isomorphic to $\mathbb{Z}$.

(6) For a nonempty space X, the *suspension* SX is the quotient of $X \times [0,1]$ by the equivalence relation whose equivalence classes are $X \times \{0\}$, $X \times \{1\}$ and single points (x,t) with $t \neq 0, 1$.

(a) Show that SX is path connected.

(b) Show that if X is path connected, then SX is simply connected.

(7) Let X be the space of all functions $[0,1] \to \mathbb{R}$, endowed with the sup-metric:

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Let $Y \subset X$ be the subspace of continuous functions. Show that Y is closed in X.