DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
AUGUST 2016

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correct.
PhD: 75% with at least three substantially correct.

1. Consider integrating
$$I(f) = \int_0^1 f(x)\,dx.$$
Suppose we cannot compute $f(x)$ directly, but instead we can compute $g(x)$ where $|g(x) - f(x)| < \epsilon$ for all $x \in [0, 1]$.
   (a) On the interval $[a, b]$, $0 \le a < b \le 1$, obtain an error estimate for the trapezoidal approximation
$$\int_a^b f(x)\,dx - \frac{b-a}{2}(g(a) + g(b)).$$
   (b) Using $g(x)$ write down a composite trapezoidal rule approximation for $I(f)$ with evenly spaced nodes $0 = x_0 < x_1 \cdots < x_n = 1$. Give an error bound for the composite rule.

2. Let
$$A = \begin{bmatrix} 10^{-20} & 2 \\ 1 & 3 \end{bmatrix}.$$
   (a) Compute the LU decomposition of A in exact arithmetic.
   (b) Compute the LU decomposition in finite precision floating-point arithmetic, assuming 15 decimal digits of accuracy. (Namely, at this precision $1 \oplus 10^{-16} = 1$, but $10^{-16} \neq 0$.)
   (c) Compare the two results.

3. Find a polynomial p of minimal degree satisfying
$$p(x_1) = y_1, \qquad p'(x_2) = y_2 \qquad p(x_3) = y_3.$$
Under what conditions is the solution unique?

4. Consider the numerical solution of $y' = f(y)$ with a scheme of the form
$$y_{n+1} = y_n + h\Big[a_1 f\big(y + hb_1 f(y)\big) + a_2 f\big(y + hb_2 f(y)\big)\Big].$$
   (a) Show that the choices $a_1 = 1, b_1 = 1/2, a_2 = 0, b_2 = 0$ give a second-order scheme.
   (b) Show that it is impossible to get a higher order scheme for general f for any choice of a_i and b_i.

5. Replace the true derivative with a constant value d in Newton's method to obtain a scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{d}$$

   (a) For what values of d, will this method be locally convergent?
   (b) Find the convergence order, and the rate if linearly convergent.
   (c) Is there any value of d what would lead to quadratic convergence?

6. For function $\sin(\pi x)$,

   (a) Find the value of a which solves the following optimization problem:

$$\min_a \int_{-1}^{1} (\sin(\pi x) - ax)^2 \ dx$$

   (b) Let $\hat{f}(x)$ be a polynomial with degree less than or equal to $n > 1$, which solves the minimization problem:

$$\min_{p(x) \in \mathbf{P_n(x)}} \int_{-1}^{1} (\sin(\pi x) - p(x))^2 \ dx$$

   Prove that $\hat{f}(x)$ is an odd function.

7. Given a vector norm $\|\cdot\|$ for the space $\mathbb{R}^n$, the induced matrix norm for an n-by-n matrix A is defined as

$$\|A\| = \max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}.$$

   For a non-singular real matrix A,

   (a) The condition number $\kappa(A) \doteq \|A\| \cdot \|A^{-1}\|$. Show that $\kappa(A) \geq 1$.
   (b) Find $\kappa(A)$ for an orthogonal matrix A, when the Euclidean norm is used.
   (c) Consider the linear system $Ax = b$ and its perturbed version $(A + \delta A)x = b + \delta b$. Show that

$$\frac{\|\delta b\|}{\|b\|} \leq \kappa(A) \frac{\|\delta A\|}{\|A\|}.$$