DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC QUALIFYING EXAM – APPLIED MATHEMATICS

Wednesday, August 31, 2016
10:00AM – 1:00PM
LGRT 1634

Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially complete solutions.

1. Suppose that a linear system with state $y(t) \in \mathbb{R}^1$ is governed by the ODE

$$\frac{dy}{dt} = -\alpha y \, + \, F \, \sin(\omega t)$$

where α is a decay rate, and the sinusoidal forcing has amplitude F and frequency ω.
(a) Solve this equation for arbitrary α, F, ω (all assumed to be positive constants), and show that the solution consists of a unique periodic solution and a transient.
(b) Determine how the magnitude of the non-transient response of the system (that is, the amplitude of the periodic part of the solution) depends on the system's parameters α, F and ω.

2. Consider the nonlinear oscillator, having coordinate $x(t) \in \mathbb{R}^1$, governed by

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \frac{dV}{dx} \, = \, 0 \, ,$$

where $\gamma > 0$ is a friction coefficient, and the potential for the restoring force is $V(x) = x^3/3 - x$.
(a) In the conservative case, when $\gamma = 0$, convert this equation into a first-order system in the usual way, determine its equilibrium points, and sketch its phase portrait.
(b) In the slightly dissipative case, when γ is small, sketch the phase portrait.
(c) Discuss (with justification) the stability of the equilibria in both cases (a) and (b).

3. Using separation of variables treat the pinned square vibrating membrane described by the equation

$$u_{tt} - c^2 \big(u_{xx} + u_{yy} \big) = 0 \, , \qquad 0 < x < 1, \quad 0 < y < 1, \quad t > 0 \, ,$$

with the (homogeneous) boundary conditions

$$u(x, 0, t) = 0 \, , u(x, 1, t) = 0 \, , u(0, y, t) = 0 \, , u(1, y, t) = 0 \, ,$$

and the initial data: $u(x, y, 0) = 0$, $u_t(x, y, 0) = 1$.
That is, determine the solution as a Fourier series and calculate its coefficients.

4.(a) Solve, using the method of characteristics, the Burgers equation

$$u_t + uu_x = 0\,, \qquad -\infty < x < \infty\,, \quad t > 0\,,$$

with initial data $u(x,0) = \sin(kx)$ at $t = 0$.
(b) What are the limitations of these solutions? How does their behavior depend on k? Draw the solution $u = u(x,t)$ at some typical time instances $0 < t_1 < t_2 < t_3$ (say) to illustrate the key features of these nonlinear waves.

5. Consider the homogeneous heat equation on the semi-infinite line forced by an *inhomogeneous* boundary condition; namely, solve the initial-boundary value problem:

$$u_t \;-\; u_{xx} \;=\; 0\,, \qquad \text{on} \;\; 0 < x < +\infty\,,\; t > 0\,,$$

subject to $u(0,t) = \phi(t)$ and $\lim_{x\to+\infty} u(x,t) = 0$; for definiteness, impose the initial condition $u(x,0) = 0$, and suppose that $\phi(0) = 0$. Obtain the solution as a Fourier integral by applying Fourier sine-transform to the PDE and carefully including the boundary condition in this calculation; the transform formulas are

$$F(k) = \int_0^{+\infty} f(x)\,\sin kx\,dx\,, \qquad f(x) = \frac{2}{\pi}\int_0^{+\infty} F(k)\,\sin kx\,dk\,.$$

6. Let $u = u(x,y)$ be the solution to the Poisson equation

$$u_{xx} + u_{yy} \;=\; f(x,y) \quad \text{in the unit disk} \;\; x^2 + y^2 < 1$$

with $u = g(x,y)$ on the boundary circle $x^2 + y^2 = 1$. Suppose that the given data, f and g, satisfy $|f(x,y)| \le M$ and $|g(x,y)| \le N$ for all relevant points, where M and N are some positive constants. Show, using the maximum principle, that the solution satisfies

$$|u(x,y)| \;\le\; N \;+\; \frac{M}{4}\big(\,1 - x^2 - y^2\,\big) \quad \text{throughout the domain} \;\; x^2 + y^2 < 1\,.$$

7. Consider the system

$$\frac{dx}{dt} = y + 0.01x(x^2 + y^2)\,, \qquad \frac{dy}{dt} = -x + 0.03y(x^2 + y^2)\,.$$

(a) Formulate the linearized system around the equilibrium point $(0,0)$ and completely describe its behavior.
(b) How does the nonlinear system itself behave as $t \to +\infty$ and how does its asymptotic behavior compare to that of the linearized system? Display a Lyapunov function that justifies the claimed behavior of the nonlinear system.