NAME:

Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question and on the blank page after each question.

1. Let $f : [0, 1] \to \mathbb{C}$ be a **continuous** function. Provide a proof for the fact that

$$\lim_{n\to\infty} \int_0^1 e^{-2\pi nx} f(x)\, dx = 0.$$

Either by proof or example, determine whether there can be an f as above, and such that

$$\left| \int_0^1 e^{-2\pi nx} f(x)\, dx \right|^2 \geq \frac{1}{n}, \quad \forall\, n \in \mathbb{N}.$$

Hint: Consider the Hilbert space $L^2(0, 1)$ with the usual product.

2. Define what it means for a function $f : [0, 1] \to \mathbb{R}$ to be of bounded variation. Then, show that if f is a bounded, nondecreasing, measurable function in $[0, 1]$, then it must be of bounded variation.

3. Let $K \in \mathcal{S}(\mathbb{R}^1)$ and $\phi \in L^1(\mathbb{R}^1) \cap L^\infty(\mathbb{R}^1)$ be such that

$$K * \phi \equiv 0 \text{ in } \mathbb{R}^1.$$

Assuming that the Fourier transform of K is never zero, prove that $\phi = 0$ a.e.

4. Let $f : \mathbb{R}^d \to \mathbb{R}$ be measurable, non-negative. Define $\lambda_f : (0, \infty) \to \mathbb{R}$ by

$$\lambda_f(\alpha) := m(\{x : f(x) > \alpha\}), \quad (m = \text{Lebesgue measure}).$$

Prove that $\lambda_f(\cdot)$ is Lebesgue measurable and that (allowing for $\infty = \infty$)

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_0^\infty \lambda(\alpha)\, d\alpha.$$

5. Let μ be a finite measure on the Borel sets of $X = [0, 1]$ such that $\mu(\{x\}) = 0$ for all $x \in X$. Prove that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\mu(A) \leq \epsilon$ for all intervals $A \subset X$ contained in $(\frac{1}{2} - \delta, \frac{1}{2} + \delta)$.

6. Given $f : \mathbb{R} \to \mathbb{R}$ define $f_h(x) = f(x - h)$.

   (a) Show that if f is continuous with compact support then $\lim_{h\to 0} \| f_h - f \|_\infty = 0$.

   (b) Show that if $f \in L^p(\mathbb{R})$ with $1 \leq p < \infty$ then $\lim_{h\to 0} \|f_h - f\|_p = 0$.

   (c) Prove or disprove by a counterexample: if $f \in L^\infty(\mathbb{R})$ then $\lim_{h\to 0} \|f_h - f\|_\infty = 0$.

7. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $f_n, f, g_n, g : X \to R$ for $n \in N$ be measurable functions. Suppose that $f_n \to f$ and $g_n \to g$ in measure as $n \to \infty$.

   (a) Prove that $f_n + g_n \to f + g$ in measure as $n \to \infty$.

   (b) Assume that that $\mu(X) \leq \infty$. Prove that $f_n g_n \to fg$ in measure as $n \to \infty$.

   (c) If $\mu(X) = \infty$, prove or disprove by a counterexample: $f_n g_n \to fg$ in measure as $n \to \infty$.

8. Consider Lebesgue measure on Borel sets of $(0, \infty)$. Prove that for every $f \in L^2(0, \infty)$

   (a) The inequality $\left|\int_0^x f(x)dx\right|^2 \leq 2\sqrt{x} \int_0^x \sqrt{s}|f(s)|^2 ds$ holds for all $x \in (0, \infty)$.
   (b) The inequality $\|F\|_2 \leq 2\|f\|_2$ where $F(x) = \frac{1}{x} \int_0^x f(s)ds$.

   Hint: for part a), consider using Hölder's or Cauchy's inequality.