DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - PROBABILITY
September 4, 2015

Work all problems. Show all work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Let Y_1 and Y_2 be independent and identically distributed random variables with the following density:

$$f(y) = \begin{cases} \lambda e^{-\lambda y}, & y \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

You are given that the moment generating function of a random variable Y is the following:

$$M_Y(t) = E[e^{tY}].$$

   (a) (7 points) Find the moment generation function of Y_1 .

   (b) (7 points) Use the moment generating function to show that $Z = Y_1 + Y_2$ has density

$$f(z) = \begin{cases} \lambda^2 z e^{-\lambda z}, & z \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

   (c) (7 points) Suppose $\lambda = 1$. Let $c > 0$. Show that the density of $Y_1|Y_1 > c$ is e^{-y+c} for $y > c$, and 0 elsewhere.

   (d) (7 points) Suppose $\lambda = 1$. Let $c > 0$. Find $E[Y_1|Y_1 > c]$.

2. Let Z be a standard normal random variable and let $X_1 = Z$ and $X_2 = Z^2$.

   (a) (7 points) Find $E(X_1)$ and $E(X_2)$.

   (b) (7 points) Find $E(X_1X_2)$.

   (c) (7 points) Find $Cov(X_1, X_2)$.

   (d) (7 points) Are X_1 and X_2 independent? Why or why not?

3. Suppose there is a music festival consisting of 6 concerts: C_1, C_2, C_3, C_4, C_5, and C_6. There is a separate ticket package available for each possible subset of concerts one could attend.

   (a) (7 points) How many ticket packages are there?

   (b) (7 points) Suppose a ticket package is selected completely at random from among the available packages. What is the probability that it includes concert C_1?

   (c) (7 points) Suppose the number of people who attend the festival, X, is distributed $Poisson(\lambda)$, and the number who buy each ticket package is given by a multinomial distribution with parameters $(X, p_1, ...p_k)$, where k is the number of packages. Find the marginal distribution of the number of people Y_i who purchase package i, $i \in 1 : k$.

4. The probability that a child has blue eyes is 1/4. Assume independence between children. Consider a family with three children.

   (a) (7 points) If it is known that at least one child has blue eyes, what is the probability that at least two children have blue eyes?

   (b) (7 points) If it is known that the youngest child has blue eyes, what is the probability that at least two children have blue eyes?

5. (9 points) Let X and Y be independent and suppose that each has a $Uniform(0, 1)$ distribution. Let $Z = min\{X, Y\}$. Find the density $f_Z(z)$ for Z. *Hint: It might be easier to first find* $P(Z > z)$.