DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERICAL ANALYSIS EXAM
SEPTEMBER 2015

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correcct

PhD: 75% with at least three substantially correct.

1. Let A be a nonsingular matrix of order n and X_k, $k = 0, 1, ...$, be a sequence of matrices of order n satisfying
$$X_{k+1} = X_k + X_k(I - AX_k).$$
   (a) Show that the inverse of A, A^{-1}, is a fixed point of the above iteration.

   (b) Assume that $\|I - AX_0\| < 1$, show that $E_k = I - AX_k$ converges quadratically to zero. (Hint: Multiply the above equation by A and subtract both sides from I.)

   (c) What is X_k converging to?

2. For function $\sin(\pi x)$,

   (a) Find the value of a which solves the following optimization problem:
$$\min_a \int_{-1}^{1} (\sin(\pi x) - ax)^2 \ dx$$
   (b) Let $\hat{f}(x)$ be a polynomial with degree less than or equal to $n > 1$, which solves the minimization problem:
$$\min_{p(x)\in \mathbf{P_n(x)}} \int_{-1}^{1} (\sin(\pi x) - p(x))^2 \ dx$$
   Prove that $\hat{f}(x)$ is an odd function.

3. Find a polynomial $p(x)$ of degree ≤ 2 that satisfies
$$p(x_0) = y_0, \ \ p'(x_0) = y_0', \ \ p'(x_1) = y_1'.$$
   Give a formula in the form
$$p(x) = y_0 l_0(x) + y_0' l_1(x) + y_1' l_2(x).$$

4. Suppose that $A \in \mathbb{R}^{2\times 2}$ has non-zero entries on the diagonal. Show that the Jacobi Iteration converges for all starting guesses if and only if the Gauss-Seidel iteration converges for all starting guesses.

5. Consider the numerical approximation of

$$\int_{-2}^{2} \exp(-x^4 - (x-1)^2)\, dx$$

using the composite trapezoidal rule with n equal intervals of length $h = 4/n$.

(a) Bound the error in terms of h.

(b) What h should we choose to guarantee an error less than 10^{-3}. Give a numerical value.

6. Suppose that we attempt to find x^* satisfying $f(x^*) = 0$ using the Newton-like iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + \delta},$$

where $\delta \in \mathbb{R}$ is fixed. Assume that $f'(x^*) \gg \delta$ and find the order of convergence and rate of convergence, if appropriate.

7. Consider the solution of $y' = f(y)$ using the integrator

$$y_{n+1} = \frac{3}{2} y_n - \frac{1}{2} y_{n-1} + (h/2 + Cf'(y_n))f(y_n).$$

(a) Show that the scheme satisfies the root condition.

(b) Find C so that the truncation error has maximal order.