COMPLEX ANALYSIS QUALIFYING EXAM
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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- Each problem is worth 10 points.
- Passing Standard: Do 8 of the following 10 problems and
  - Master's level: 45 points with 3 questions essentially complete
  - Ph. D. level: 55 points with 4 questions essentially complete
- Justify your reasoning!

1. (a) Write down Cauchy–Riemann equations in polar coordinates. (b) Use part (a) to show that the main branch of Log is a holomorphic function. Here we define $\mathrm{Log}(re^{i\theta}) := \ln(r) + i\theta$ for $r > 0$, $-\pi < \theta < \pi$.

2. Prove the Schwarz reflection principle. Namely, let $\Omega \subset \mathbb{C}$ be an open set symmetric under complex conjugation. Let

$$\Omega^+ = \Omega \cap \{z \,|\, \mathrm{Im}(z) > 0\}, \quad \Omega^- = \Omega \cap \{z \,|\, \mathrm{Im}(z) < 0\}, \quad I = \Omega \cap \mathbb{R}.$$

Suppose $f(z)$ is a holomorphic function in Ω^+ which extends continuously to $\Omega^+ \cup I$. Then $f(z)$ can be extended to a holomorphic function in Ω.

3. Find a holomorphic bijection between the region

$$\{z \;:\; |z| < 2,\ \mathrm{Im}(z) > 1\}$$

and the region

$$\{z \;:\; |z| < 2,\ \mathrm{Im}(z) < 1\}.$$

4. (a) Determine the number of zeroes of

$$z^5 - z^4 + 2z^3 - 3z^2 - 5$$

in the disk $\{z \;:\; |z| < 3\}$.
(b) Evaluate the integral $\displaystyle\int_C \frac{z^4 - 2z^2 + z - 3}{z^5 - z^4 + 2z^3 - 3z^2 - 5}dz$, where C is the positively-oriented boundary of the disc from part (a).

5. Evaluate the integral

$$\int_0^\infty \frac{x^{1/3}}{x^2 + 9x + 8}dx$$

Justify all your steps.

6. Let z_0 be an isolated singularity of an analytic function f. Prove that if $\mathrm{Re}(f)$ is bounded from above, then z_0 is a removable singularity.

7. For each of the following functions, find all isolated singular points, classify them (into removable singularities, poles, essential singularities), and find residues at all isolated singular points:

$$\text{(a)} \quad z^2 e^{\frac{1}{z+1}}; \quad \text{(b)} \quad \cot^2(z); \quad \text{(c)} \quad \frac{z^{35}}{1 - z^{16}}.$$

8. Find all Laurent series of $f(z) = \frac{2z}{z^2-4z+3}$ centered at the origin and specify for each the largest region over which it represents the function.

9. Prove the open mapping theorem: a holomorphic non-constant function $f : \Omega \to \mathbb{C}$ is open, i.e. $f(U)$ is open for any open set $U \subset \Omega$. Here $\Omega \subset \mathbb{C}$ is a connected open set.

10. Let $F(z,w) = w^n + c_1(z)w^{n-1} + \cdots + c_n(z)w^n$, where $c_1(z), \ldots, c_n(z)$ are entire functions. Assume that the polynomial $F(0,w)$ has a unique and simple zero w_0 in the open unit disk $D := \{w \;:\; |w| < 1\}$ and $F(0,w)$ does not vanish on the boundary $\{w \;:\; |w| = 1\}$.

(a) Prove that the integral

$$\frac{1}{2\pi i}\int_{|w|=1} \frac{\frac{\partial F}{\partial w}(z,w)}{F(z,w)}dw$$

is constantly equal to 1, for z in some non-empty connected open neighborhood U of 0 in the complex plane.

(b) Prove that the integral

$$\frac{1}{2\pi i}\int_{|w|=1} w\frac{\frac{\partial F}{\partial w}(z,w)}{F(z,w)}dw$$

is a well defined holomorphic function $\varphi(z)$ of z in some non-empty connected open neighborhood U of 0 in the complex plane. Moreover, $\varphi(0) = w_0$ and $F(z,\varphi(z)) = 0$, for all $z \in U$.