DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
ADVANCED EXAM - DIFFERENTIAL EQUATIONS

Wednesday September 2, 2015
10:00AM – 1:00PM

Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions, including one from the ODE part (Questions 1-3) and one from the PDE part (Questions 4-7).

(1) Consider the linear system $\dot{x} = Ax$ with coefficient matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

We say that $x(t)$ grows linearly if $\lim_{t\to+\infty} |x(t)|/t = c > 0$ and superlinearly if $\lim_{t\to+\infty} |x(t)|/t = +\infty$. Find all initial conditions $x(0)$ such that their solutions $x(t)$ are (a) bounded; (b) grow linearly; (c) grow superlinearly.

(2) a) Consider a planar autonomous system $\dot{x} = f(x)$, for $x = (x_1, x_2) \in \mathbb{R}^2$, and assume that the vector field $f(x)$ is divergence free, that is,

$$\text{div } f \doteq \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 \qquad \text{identically in } \mathbb{R}^2 .$$

Prove that this dynamical system has no (isolated) periodic orbits.
b) Construct an explicit vector field, $f(x)$, on $\mathbb{R}^2$ with div $f > 0$ in $|x| < 1$, and div $f < 0$ in $|x| > 1$, such that the unit circle, $|x| = 1$, is an isolated limit cycle for f.

(3) Consider the nonautonomous, nonlinear, second-order differential equation

$$\frac{d^2x}{dt^2} + (1 - \alpha \sin t) \left[\frac{dx}{dt}\right]^3 + x \, = \, 0 \, ,$$

where α is a constant satisfying $0 < \alpha < 1$. Prove that the origin, that is, $(x, \dot{x}) = (0, 0)$, is an asymptotically stable fixed point for this equation.

(4) a) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth and bounded function. Use the method of characteristics to find the solution of the following Cauchy problem,

$$\begin{aligned} u_y - xu_x &= -u, \qquad x \in \mathbb{R},\ y > 0 \\ u(x,0) &= f(x). \end{aligned}$$

What is $\lim_{y \to +\infty} u(x,y)$?

b) Let α, β, γ be real constants and $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ a smooth function. Use the method of characteristics to solve the following Cauchy problem,

$$\begin{aligned} \alpha u_{x_1} + \beta u_{x_2} + u_{x_3} &= -\gamma\, u, \qquad x_3 > 0 \\ u(x_1, x_2, 0) &= \phi(x_1, x_2). \end{aligned}$$

(5) Use d'Alembert's formula and Duhamel's principle to solve the following Cauchy initial value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = \cos x \\ u(x,0) = \sin x, \quad u_t(x,0) = 1 + x \end{cases}$$

(6) Let $\Omega \subset \mathbb{R}^2$ be a smooth domain and $u = u(x,t)$ a smooth solution to the following initial boundary value problem

$$\begin{cases} u_t = \Delta u - u^3 & \text{in} \quad x \in \Omega, t > 0 \\ u(x,0) = 0, & \text{for all} \quad x \in \Omega \\ u(x,t) = 0 & \text{for all} \quad x \in \partial\Omega, t \geq 0. \end{cases}$$

Show that $u(x,t) = 0$ for all $x \in \Omega$ and $t \geq 0$.

(7) We say that a function $u \in C^2(\overline{\Omega})$ is *subharmonic* if $-\Delta u \leq 0$ in Ω.

(a) Prove that if $u \in C^2(\bar{\Omega})$ is subharmonic then

$$u(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy \qquad \text{for all } B(x,r) \subset \Omega.$$

(b) Prove that therefore, $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.

(c) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth and convex function. Assume v is harmonic and let $u(x) := \varphi(v(x))$. Prove that u is subharmonic.

(d) Prove that $u := |Dv|^2$ is subharmonic whenever v is harmonic.