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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question and on the blank page after each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.

2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.

3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. Let m denote Lebesgue measure on $[0, 1]$. For $n \in \mathbb{N}$ define the intervals $A_n = (\frac{1}{n+1}, \frac{1}{n}]$. For $\alpha \in \mathbb{R}$ define the function $f : [0, 1] \mapsto \mathbb{R}$ by $f(0) = 0$ and

$$f(x) = \sum_{n=1}^{\infty} n^{\alpha} \, 1_{A_n}(x) \quad \text{for } 0 < x \leq 1.$$

Find the values of $\alpha \in \mathbb{R}$ for which f is integrable — that is, for which $f \in L^1([0, 1], m)$. Then **prove** that f is integrable for these values of $\alpha \in \mathbb{R}$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n, n \in \mathbb{N}\}$ be a sequence of real-valued functions in $L^2(X, \mu)$ such that

$$\sup_{n \in \mathbb{N}} \int_X f_n^2 \, d\mu \leq C < \infty$$

for some positive constant C. Prove that $\lim_{n\to\infty} f_n(x)/n = 0$ for μ-almost every $x \in X$. [**Hint.** Consider the sequence of functions $g_n = \sum_{k=1}^n f_k^2/k^2$ for $n \in \mathbb{N}$.]

3. (a) Let g be a function that maps $[0,1]$ into $\mathbb{R}$. Define the concept "g is of bounded variation on $[0,1]$."

Define the function $f : [0,1] \mapsto \mathbb{R}$ by $f(0) = 0$ and

$$f(x) = x^2 \cos(1/x^2) \quad \text{for } 0 < x \leq 1.$$

(b) Prove that f is differentiable at each $x \in (0,1)$ and has a right hand derivative at $x = 0$.

(c) Prove that f is not of bounded variation on $[0,1]$ by considering the sequence of partitions indexed by $n \in \mathbb{N}$

$$\mathcal{P}_n = \left\{0, \left(\frac{2}{2n\pi}\right)^{1/2}, \left(\frac{2}{(2n-1)\pi}\right)^{1/2}, \ldots, \left(\frac{2}{3\pi}\right)^{1/2}, \left(\frac{2}{2\pi}\right)^{1/2}, 1\right\}.$$

4. (a) Give the definition of the outer measure m^* that arises in the construction of Lebesgue measure m on $\mathbb{R}$. Some authors refer to the outer measure as the exterior measure.

   (b) Prove that $m^*(A + s) = m^*(A)$ for any subset A of $\mathbb{R}$ and any $s \in \mathbb{R}$.

   (c) Prove that for any nonnegative, Borel-measurable function f mapping $\mathbb{R}$ into $\mathbb{R}$ and for any $t \in \mathbb{R}$

$$\int_{\mathbb{R}} f(x - t)\, dm(x) = \int_{\mathbb{R}} f(x)\, dm(x).$$

   [**Hint.** First prove (c) for 1_B, where B is a Borel subset of $\mathbb{R}$.]

5. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. Let $f : X \mapsto [0, \infty)$ and $g : Y \mapsto [0, \infty)$ be measurable functions. In this problem the space $X \times Y$ is equipped with the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$.

   (a) Prove that f and g are both measurable functions on $X \times Y$ and that $h(x, y) = f(x)g(y)$ is also a measurable function on $X \times Y$.

   (b) Assume that $f \in L^1(X, \mu)$ and $g \in L^1(Y, \nu)$. By applying an appropriate theorem, prove that $h \in L^1(X \times Y, \mu \times \nu)$.

6. Let E be a nonempty closed and convex set in a Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$. Prove that there exists a unique element $x_0 \in E$ which minimizes $\|x\|$ on E; that is, $\|x_0\| = \inf_{x\in E} \|x\|$. [**Hint.** Use the parallelogram law $\|y+z\|^2 + \|y-z\|^2 = 2\|y\|^2 + 2\|z\|^2$ for all y and z in $\mathcal{H}$. Apply the parallelogram law twice, first to an appropriate sequence in order to prove the existence of x_0 and then to prove the uniqueness of x_0.]

7. (a) Let A be a proper subset of $[0, 1]$ which is measurable. Consider the limit

$$\lim_{n\to\infty} \int_A \cos(2\pi nx)\, dx = 0 = \lim_{n\to\infty} \int_A \sin(2\pi nx)\, dx.$$

Either prove this limit or invoke a theorem that implies this limit. [**Hint.** Use the fact that $1_A \in L^2([0, 1], m)$, where m denotes Lebesgue measure on $[0, 1]$.]

(b) One can prove that the series $\sum_{n=1}^{\infty} [\sin(2\pi nx)]/\sqrt{n}$ converges for all $x \in [0, 1]$ (you do not have to prove this). Could this series be the Fourier series of a function in $L^2([0, 1], m)$? Explain your answer.

8. (a) Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < \infty$. Let $\{f_n, n \in \mathbb{N}\}$ and f be measurable functions that map X into $\mathbb{R}$. Prove that if $f_n \to f$ μ-a.e., then $f_n \xrightarrow{\mu} f$ (convergence in measure). [**Hint.** One way to prove this is to use Egoroff's Theorem.]

   (b) Show that Egoroff's Theorem fails for the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$, where m denotes Lebesgue measure.