DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ADVANCED EXAM — ALGEBRA

SEPTEMBER 2015

Passing Standard: To pass the exam it is sufficient to solve five problems including a least one problem from each of the three parts. Show all your work and justify your answers carefully.

1. Group theory and representation theory

1. Classify all groups of order 18 and order 20.

2. Let k be a field of characteristic p and G a finite group such that $|G|$ is divisible by p. Set

$$\epsilon = \sum_{g \in G} g \in k[G].$$

Show that the $k[G]$-submodule $k[G]\epsilon$ of the free module $k[G]$ is not a direct summand.

3.

(1) Is there a group G with $\mathbf{C}[G]$ isomorphic to $\mathbf{C} \times \mathbf{C} \times M_2(\mathbf{C})$?

(2) Is there a group G with $\mathbf{C}[G]$ isomorphic to $\mathbf{C} \times \mathbf{C} \times M_3(\mathbf{C})$?

2. Commutative algebra

4. Let A be a ring and $I \subset A$ is a proper ideal. Prove that the radical $\sqrt{I}$ is the intersection of all prime ideals containing I.

5. Prove that every prime ideal in $\mathbf{Z}[x]$ can be generated by at most two elements.

6. Let k be a field and $R = k[x,y]_{(x,y)}/(x^2, xy)$. (That is, R is the quotient of the localization of $k[x,y]$ at the prime ideal (x,y) by the ideal generated by x^2 and xy.) Prove that R is equal to its ring of fractions and that $\dim(R) = 1$.

3. Field theory and Galois theory

7. Find the Galois group of the polynomial $x^4 + 2x^2 + 4$ over $k = \mathbb{Q}$.

8. How many roots of the polynomial $x^{12} + x^8 + x^4 + 1$ lie in the field $\mathbf{F}_{121}$ of order 121?

9. Determine all possible Galois groups of the splitting field of the polynomial $f_a(x) = x^4 - a$ for $a \in \mathbf{Q}$.