DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
AUGUST 2014

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correct.

PhD: 75% with at least three substantially correct.

1. Find coefficients a_0, a_1, a_2 and a_3 and the node x_0 in such a way that

$$\int_{-1}^{1} f(x)dx \approx a_0 f(-1) + a_1 f(-x_0) + a_2 f(x_0) + a_3 f(1)$$

has the highest degree of accuracy.

2. Consider the data $(0, 0)$, $(1/3, 1/2)$ and $(1, 1)$.

   (a) Find the cubic spline that interpolates these data and satisfies the natural boundary conditions.

   (b) Determine the approximate value $f(1/4)$ using the spline.

3. Given an ODE initial-value problem

$$\frac{dy}{dx}(x) = f(x, y(x)),$$

with initial data $y(x_0) = y_0$, we would like to solve this initial value problem at points $x_n = nh, n = 0, ..., N$ where $h = x_n - x_{n-1}$ for all n with a method of the form

$$y_{n+1} = y_n + h[b_1 f(x_n, y_n) + b_2 f(x_{n-1}, y_{n-1}) + b_3 f(x_{n-2}, y_{n-2})].$$

   (a) Find b_1, b_2, b_3 for the above method which gives the highest order local truncation error.

   (b) Give the local truncation error of the method obtained.

   (c) State the order of the method obtained.

4. For the linear system

$$\begin{pmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

   (a) Write down the Gauss-Seidel method for the iterative solution of the above problem.

   (b) Give the iteration matrix associated with the above method and compute the spectral radius.

   (c) What can you say about the convergence of the above method and its rate of convergence.

5. Consider the step function $f(x)$:

$$f(x) = \begin{cases} 1, & x \in [-1, 0], \\ -1, & x \in (0, 1]. \end{cases}$$

   (a) Compute the constant, linear, and quadratic least square approximate to $f(x)$.

(b) Also compute the L^2 errors of these three least square approximates.

6. Using the Secant method to find a root α for function $f(x)$,

   (a) Prove that

   $$x_{k+1} - \alpha = \frac{f[x_k, x_{k-1}, \alpha]}{f[x_k, x_{k-1}]}(x_k - \alpha)(x_{k-1} - \alpha),$$

   where the two $f[\ldots]$ are the Newton Divided Differences.

   (b) Assuming

   $$M = \frac{1}{2}\frac{\sup_{x_2} |f''(x_2)|}{\inf_{x_1} |f'(x_1)|} > 0,$$

   show that

   $$|x_k - \alpha| \le \frac{z_k}{M},$$

   where the sequence z_k is defined by $z_{k+1} = z_k z_{k-1}$.

7. Define function $f(x)$ as

   $$f(x) = \begin{cases} \sin x, & x \in [0, 1], \\ \cos x, & x \in (1, 2\pi]. \end{cases}$$

   (a) Find the second order polynomial interpolation to $f(x)$, with interpolation points $\{0, \pi, 2\pi\}$. Also compute the maximum error of the above interpolation.

   (b) Prove that the maximum error for polynomial interpolation of any degree will be NOT less than $|\sin 1 - \cos 1|/2$.