COMPLEX ANALYSIS BASIC EXAM
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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- Each problem is worth 10 points.
- Passing Standard: **Do 8 of the following 10 problems**, *and*
  - Master's level: 45 points with three questions essentially complete
  - Ph. D. level: 55 points with four questions essentially complete
- Justify your reasoning!

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1. Determine the Laurent series of $\frac{1}{(z-1)(z-2)}$ for the region $1 < |z| < 2$ **and** for the region $|z| > 2$.

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2. Let U be a connected open set, and let D be an open disk whose closure is contained in U. Let f be analytic on U and not constant. Assume that the absolute value $|f(z)|$ is constant on the boundary of D. Prove that f has at least one zero in D. *Hint: Consider $g(z) := f(z) - f(z_0)$ with $z_0 \in D$.*

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3. Evaluate the integrals $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 2x + 2} dx$ and $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 - 2x + 2} dx$. Show the contour and prove all estimates you use.

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4. Let $f(z)$ be an entire function. Suppose there exists a positive integer n such that $|\mathrm{Re}(f(z))| \le |z|^n$ for all $z \in \mathbf{C}$. Show that $f(z)$ is a polynomial of degree at most n.

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5. Show that the following series defines a meromorphic function on $\mathbf{C}$ and determine the set of poles and their orders.

$$\frac{1}{z} + \sum_{n \neq 0, n=-\infty}^{\infty} \left[\frac{1}{z-n} - \frac{1}{n} \right].$$

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6. Let $P(z)$ be a non-constant polynomial.

(a) Suppose that the roots of $P(z)$ all lie on the same side of a straight line L. Show that the roots of the derivative $P'(z)$ also lie in the same infinite region.

Hint: Reduce to the case where L is the imaginary axis, and then use an explicit computation of the logarithmic derivative of $P(z)$ in this case.

(b) Deduce that if the roots of $P(z)$ all lie inside a circle C, then the roots of $P'(z)$ also lie inside C.

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7. Evaluate the integral $\displaystyle\int_{|z|=5} e^{e^{(1/z)}}\,dz$.

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8. (a) Give the precise statement of the Riemann Mapping Theorem (including uniqueness).

(b) Denote by $\mathbb{D} = \{|z| < 1\}$ the open unit disk. Determine **all** conformal maps φ taking $\{z \in D : \mathrm{Re}(z) > 0\}$ onto $\mathbb{D}$ such that $\varphi(\sqrt{2}-1) = 0$.

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9. Let $f_1(z)$ be an analytic function on the open unit disk $\mathbb{D} = \{|z| < 1\}$ such that $f_1(0) = 0$ and $|f_1(z)| < 1$ for all $z \in \mathbb{D}$. For every integer $n \geq 1$, define $f_{n+1}(z) = f_n(f_1(z))$.

Suppose that the limit $g(z) := \lim_{n\to\infty} f_n(z)$ exists for every $z \in \mathbb{D}$. Show that either $g(z)$ is identically zero or $g(z) = z$ for every $z \in \mathbb{D}$.

Note: We are not assuming that $g(z)$ is analytic on $\mathbb{D}$.

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10. Compute the integral

$$\int_0^{\pi} \frac{d\theta}{3 + 2\cos(\theta)}.$$
