University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
For August 2014

Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. Prove or Disprove:

   (a) Let M be a smooth oriented manifold, ψ_t, $t \in \mathbb{R}$, be a smooth flow generated by a smooth vector field. Then for any t, $\psi_t : M \to M$ is orientation-preserving.

   (b) There exists an $n \geq 1$, such that the tautological line bundle E over the real projective space $\mathbb{R}P^n$ is trivial. (Here by definition, the fiber of E over a point $p = \mathbb{R}x \in \mathbb{R}P^n$, where $0 \neq x \in \mathbb{R}^{n+1}$, is the 1-dimensional subspace $E_p = \mathbb{R}x \subset \mathbb{R}^{n+1}$. Note that E may be regarded as a sub-bundle of the trivial $\mathbb{R}^{n+1}$ bundle over $\mathbb{R}P^n$.).

2. Let $M = S^2 \times S^2$, and let τ be the free involution on M which sends (x, y) to $(-x, -y)$, $x, y \in S^2 \subset \mathbb{R}^3$.

   (a) Compute the De Rham cohomology groups of M via the Mayer-Vietoris sequence.

   (b) Determine the induced action of τ on the De Rham cohomology groups of M.

3. Let M be a compact smooth n-manifold (without boundary) and $f : M \to \mathbb{R}$ be a smooth function. Show that for any interval $[a, b] \subset \mathbb{R}$ which does not contain any critical values of f, $f^{-1}([a, b])$ is diffeomorphic to $f^{-1}(a) \times [a, b]$. Furthermore, suppose $n = 2$ and f has only two critical points $p, q \in M$ such that there are local coordinates x, y near p, q in which f is given by $f(x, y) = f(p) + x^2 + y^2$ and $f(x, y) = f(q) - x^2 - y^2$ respectively. Show that in this case M must be homeomorphic to the 2-sphere S^2.

4. Let G be a finite group acting smoothly on a smooth n-manifold M. For any $p \in M$, we let G_p be the subgroup of G defined by $G_p = \{g \in G | g \cdot p = p\}$. Show that for any $p \in M$, there exists a local chart (U, ϕ) centered at p, such that (1) $g \cdot U = U$ for all $g \in G_p$, (2) there is a linear action of G_p on $\mathbb{R}^n$ such that $\phi : U \to \mathbb{R}^n$ is G_p-equivariant, i.e., for any $q \in U$, $g \in G_p$, $\phi(g \cdot q) = g \cdot \phi(q)$. (Hint: consider the exponential map $\exp_p$.)

5. Consider smooth vector fields $U = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}$, $V = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$ on $\mathbb{R}^3$.

   (a) Show that $[U, V] = 0$.

   (b) Find a local coordinate chart (u, v, w) centered at the origin such that $U = \frac{\partial}{\partial u}$ and $V = \frac{\partial}{\partial v}$.

6. Consider the Riemannian metric $g = e^{2u}(dx^2 + dy^2)$ on an open subset $M \subset \mathbb{R}^2$ where $u\colon M \to \mathbb{R}$ is a smooth function. Calculate the Levi-Civita connection, the curvature, and the geodesic equations for this metric. Specialize your formulas to the case when $u = -\ln y$ and M is the upper half plane $y > 0$. Find at least one non-trivial geodesic in this case.

7. Let (M, g) be an oriented connected Riemannian manifold.

   (a) Define the Hodge Laplace operator $\triangle$ on functions $f \in C^\infty(M)$.

   (b) Show that for M compact the only solutions to $\triangle f = 0$ are constant functions.

   (c) Show that for M compact a necessary condition for the solvability of $\triangle f = h$ is $\int_M h \, dvol_g = 0$.

8. Let $L \to M$ be a complex line bundle over a real manifold M. For a connection ∇ on L we denote the curvature 2-form by $R^\nabla \in \Omega^2(M, \mathbb{C})$.

   (a) Show that R^∇ is a closed form.

   (b) Show that the DeRham cohomology class of R^∇ does not depend on the connection ∇.