UMass Amherst Algebra Advanced Exam
Friday August 29, 2014, 10AM – 1PM.

Instructions: To pass the exam it is sufficient to solve five problems including a least one problem from each of the three parts. Show all your work and justify your answers carefully.

1. Group theory and representation theory

Q1.

(a) Let G be a simple group of order 168. Determine the number of elements of G of order 7.
(b) Let G be a group of order 20. Suppose G contains an element of order 4 and has trivial center. Describe G in terms of generators and relations.

Q2. Let G be a finite group and p a prime dividing $|G|$. Suppose H is a subgroup of G of index p.

(a) What are the possibilities for the number of conjugate subgroups of H?
(b) Suppose in addition that p is the smallest prime dividing $|G|$. Prove that H is normal.

Q3. Let p be a prime. Let G be the subgroup of $\mathrm{GL}_3(\mathbb{F}_p)$ consisting of all matrices of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute the number of irreducible complex representations of G and their dimensions.

2. Commutative Algebra

Q4.

(a) Prove that $\mathbb{Z}[\sqrt{-2}]$ is a unique factorization domain.
(b) Prove that $\mathbb{Z}[\sqrt{-3}]$ is not a unique factorization domain.

Q5. Let R be a commutative ring with 1. Let I and J be ideals of R. Prove that the R-module $(R/I) \otimes_R (R/J)$ is isomorphic to $R/(I+J)$.

Q6. Let k be an algebraically closed field. Consider the set

$$X = \{(t^3, t^4, t^5) \mid t \in k\} \subset k^3.$$

(a) Compute generators for the ideal $I \subset k[x, y, z]$ of polynomials vanishing at each point of the set X, that is,

$$I = \{f \in k[x, y, z] \mid f(p) = 0 \text{ for all } p \in X\}.$$

(b) Determine the integral closure of the quotient ring $k[x, y, z]/I$ in its field of fractions.

3. Field theory and Galois theory

Q7. Let K be the splitting field of the polynomial $f(x) = x^4 - 2x^2 - 1$ over $\mathbb{Q}$. Determine the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

Q8. Let p be a prime. Let K be a field of order p^{28}. Determine the number of elements $\gamma \in K$ such that $K = \mathbb{F}_p(\gamma)$.

Q9. Let $\alpha \in \mathbb{C}$ be a root of the polynomial $f(x) = x^3 + 4x + 2$. For $n \in \mathbb{N}$, let $\zeta_n \in \mathbb{C}$ denote a primitive nth root of unity. Prove that α is not contained in the cyclotomic field $\mathbb{Q}(\zeta_n)$ for any n.