UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Probability
Wednesday, August 28, 2013

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Let X_1 and X_2 be independent, identically distributed random variables that have the common probability density function (p.d.f.):

$$f(x) = \begin{cases} \exp(-x), & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

   Let $Y = X_1 + X_2$ and $Z = X_1 - X_2$.

   (a) Find the joint p.d.f. of Y and Z.

   (b) Find the marginal p.d.f. of Y.

   (c) Find the conditional p.d.f. of Z, given that $Y = y$.

2. Suppose that we have a random sample $X_1, \ldots, X_n$ from a continuous distribution with cumulative distribution function(CDF) $F(x)$ and probability density function(PDF) $f(x)$.

   (a) Show that the probability that exactly r of $X_1, \ldots, X_n$ is less than or equal to x is

$$\binom{n}{r} F(x)^r (1 - F(x))^{n-r}.$$

   (b) Find the probability that at least r of $X_1, \ldots, X_n$ is less than or equal to x.

   (c) Show that the CDF of the r-th order statistic of $X_1, \ldots, X_n$ is

$$\sum_{i=r}^{n} \binom{n}{i} F(x)^i (1 - F(x))^{n-i}$$

   (d) Let $X_1, \ldots, X_n$ be the independent random variables uniformly distributed over $(0, 1)$. Show that the r-th order statistic of $X_1, \ldots, X_n$ is distributed as $Beta(r, n-r+1)$. Use the following hints :

      i. From part (c) above, the probability density function of the r-th order statistic of $X_1, \ldots, X_n$ is

$$\frac{n!}{(r-1)!(n-r)!} f(x) F(x)^{r-1} (1 - F(x))^{n-r}$$

ii. A random variable Z with a $Beta(a, b)$ distribution has the probability density function

$$f(z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} z^{a-1}(1-z)^{b-1}, \quad 0 < z < 1$$

3. Let Y_1 have an exponential distribution with mean λ and variance λ^2, and let the conditional density of Y_2 given $Y_1 = y_1$ be

$$f(y_2 \mid y_1) = \begin{cases} 1/y_1, & 0 \leq y_2 \leq y_1 \\ 0, & \text{elsewhere} \end{cases}$$

   (a) Find $E(Y_2 \mid Y_1)$.
   (b) Find $E(Y_2^2 \mid Y_1)$.
   (c) Find $Var(Y_2 \mid Y_1)$.
   (d) Find $E(Y_2)$; that is, the unconditional mean of Y_2 (you may use results from previous parts).
   (e) Find $Var(Y_2)$; that is, the unconditional variance of Y_2 (you may use results from previous parts).

4. Central Limit Theorem and its usage for Binomial Data

   (a) State a Central Limit Theorem for a sequence of i.i.d. random variables.
   (b) Let $X \sim Binom(n, p)$ be a binomial random variable with n trials and probability of success p. Show by the Central Limit Theorem that for sufficiently large n, the sampling distribution of X/n is approximately normal with mean p and standard deviation $\sqrt{p(1-p)/n}$.
   (c) About 22.9% of adults in State A are everyday smokers. Explain in details how to compute the approximate probability that between 150 and 170 of a random sample of 700 adults in State A are everyday smokers.