Complex analysis qualifying exam

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Do 8 out of the following 10 questions.

Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points with 3 questions essentially correct. To pass at the PhD level it is sufficient to have 55 points with 4 questions essentially correct.

Note: All answers should be justified carefully.

(1) (10 points) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $\frac{f(z)}{z} \to 0$ as $|z| \to \infty$. Prove that f is constant.

(2) (a) (2 points) State Rouché's theorem.

(b) (8 points) Consider the function

$$f\colon \mathbb{C} \to \mathbb{C}, \quad f(z) = z^5 + e^z + 4.$$

Let

$$\Omega = \{z = x + iy \in \mathbb{C} \mid x < 0\} \subset \mathbb{C},$$

the left half plane. Show that f has exactly 3 zeroes in Ω (counting multiplicities).

(3) (a) (5 points) Let

$$\Omega_1 = \{z = x + iy \in \mathbb{C} \mid 0 < y < 1\},$$

a horizontal strip, and

$$\Omega_2 = \{z = x + iy \in \mathbb{C} \mid x > 0 \text{ and } y > 0\},$$

the positive quadrant. Find a holomorphic bijection $f\colon \Omega_1 \to \Omega_2$.

(b) (5 points) Let

$$\Omega_3 = \{z \in \mathbb{C} \mid |z-1| < 1 \text{ and } |z-i| < 1\}$$

(a "lune"). Find a holomorphic bijection $g\colon \Omega_3 \to \Omega_2$.

(4) (a) (2 points) Let $\Omega \subset \mathbb{C}$ be an open set, $a \in \Omega$ a point, and

$$f\colon \Omega \setminus \{a\} \to \mathbb{C}$$

a holomorphic function. Define the residue of f at a.

(b) Let γ denote the circle with center the origin and radius 3, traversed once counterclockwise. Compute the following contour integrals.

i. (4 points)

$$\int_\gamma \frac{z^2}{(z-2)(z+1)^2} dz.$$

ii. (4 points)

$$\int_\gamma \frac{e^z}{\sin z} dz.$$

(5) (10 points) Compute the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{x^6+1} dx.$$

(6) Let f be a one-to-one holomorphic map from a region Ω_1 onto a region Ω_2. Assume that the closure of the disc $D := \{z : |z-z_0| < \epsilon\}$ is contained in Ω_1. Prove that the inverse function $f^{-1} : f(D) \to D$ is given by the integral formula

$$f^{-1}(\omega) = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f'(z)}{f(z)-\omega} \cdot z dz.$$

(7) Let Ω be a connected open subset of the complex plane and $f_n(z)$, $n \geq 1$, a sequence of holomorphic and nowhere vanishing functions on Ω. Assume that the sequence$f_n(z)$ converges to a function $f(z)$, uniformly on every compact subset of Ω. Prove that f is either identically zero, or never equal to zero in Ω.

(8) Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree $n > 0$. Prove that

$$\frac{1}{2\pi i} \int_C z^{n-1} |f(z)|^2 dz = a_0 \overline{a_n} R^{2n},$$

where C is the circle $|z| = R$ traversed once counterclockwise.

(9) Let C be the circle $\{|z| = 2\}$ traversed counter-clockwise. Compute $\displaystyle\int_C \frac{z^{2n} \cos(1/z)}{1 - z^n} dz$ for all integers $n \geq 2$.

(10) Prove or disprove the following statements.

(a) Let U be a simply connected open subset of the complex plane. For any two points p, q in U there exists a one-to-one holomorphic map from U onto itself such that $f(p) = q$.

(b) For any open subset W of the complex plane, any harmonic function on W is the real part of a holomorphic function on W.

(c) If f and g are meromorphic on the complex plane, then so is the composition $f \circ g$.