FALL 2013 BASIC EXAM – ADVANCED CALCULUS & LINEAR ALGEBRA
DEPARTMENT OF MATHEMATICS & STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

Provide solutions for **seven (7)** of the following ten (10) problems. Each problem is worth 10 points. To pass at the Master's level, it is sufficient to have 42 points (60%), with 3 essentially correct solutions (including at least one from each part); 53 points (75%) with at least two essentially complete solutions from each part is sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded. Be sure to show all your work.

Part I. Linear Algebra

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(1) Consider a linear operator $T : V \to V$ on a complex vector space V. Prove that if $v_1, \dots, v_n$ are eigenvectors of T with n distinct corresponding eigenvalues, then $\{v_1, \dots, v_n\}$ is linearly independent.

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(2) Find a monic polynomial $f(x)$ of degree 3 such that $f(A) = 0$ where A is the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 2 & 1 & 0 \end{pmatrix}.$$

Prove that $f(x)$ is unique.

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(3) Prove that if A is an $n \times n$ real matrix with transpose A', then A' and $A'A$ have the same range (image).

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(4) Find an orthonormal basis for the subspace $V = \{(x, y, z) \in \mathbb{R}^3 \mid 3x + y + 2z = 0\}$ of $\mathbb{R}^3$. Here orthonormal means "orthonormal relative to the usual dot product on $\mathbb{R}^3$."

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(5) Let A and B be complex $n \times n$ matrices. Prove or disprove each of the following statements.
(a) If A and B are diagonalizable, then so is $A + B$.
(b) If A and B are diagonalizable, then so is AB.
(c) If $A^2 = A$, then A is diagonalizable.
(d) If A^2 is diagonalizable, then A is diagonalizable.

Part II. Advanced Calculus

(6) Evaluate

$$\iint_D \sin(x^2+y^2)dxdy,$$

where D is the region in the First Quadrant lying between the arcs of the circles centered at 0 with radius $\sqrt{\pi/2}$ and $\sqrt{\pi}$. In other words, D is the region bounded by

$$x^2+y^2=\pi/2, x^2+y^2=\pi, x=0, y=0.$$

(7) Define a sequence of positive real numbers as follows. Suppose $x_0>0$ is a fixed positive real and for $n \geq 0$, put $x_{n+1}=(1+x_n)^{-1}$. Prove that the sequence (x_n) converges and find its limit.

(8) Consider a function f which is differentiable on $\mathbb{R}$, with $f(0)=0$ and whose derivative $f'(x)$ is an *increasing* function of x on $[0,\infty)$. Define a function g on $[0,\infty)$ by

$$g(x)=\begin{cases} f(x)/x & x>0 \\ f'(0) & x=0 \end{cases}.$$

Prove that g is an increasing function of x on $[0,\infty)$.

(9) Write down the Maclaurin series for $1/(1+x)$. By performing a suitable operation on this power series, prove the identity

$$\ln(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots.$$

Be sure to justify all steps, especially if it involves taking a limit.

(10) (a) Prove that if $f: I \to \mathbb{R}$ is uniformly continuous on an interval $I \subset \mathbb{R}$ and (a_n) is a Cauchy sequence in $\mathbb{R}$, then $(f(a_n))$ is a Cauchy sequence.

(b) Use part (a) to prove that $f(x)=1/x$ is not uniformly continuous on $(0,1]$.