University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
For August 2013

Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. Prove or disprove the following statements:
   (a) The tangent bundle $T(RP^2 \times S^1)$ is trivial.
   (b) The connected sum of RP^3 with itself is orientable.

2. Consider the smooth map $f : \mathbb{R}^3 \to \mathbb{R}^4$ given by $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Let M be the image of the restriction of f on the unit sphere $x^2 + y^2 + z^2 = 1$. Show that M is an embedded submanifold of $\mathbb{R}^4$.

3. Use the Mayer-Vietoris sequence and induction to compute the de Rham cohomology groups of the complex projective spaces CP^n.

4. Let D be the 2-dimensional smooth distribution on $\mathbb{R}^3$ spanned by vector fields $X = \frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$, $Y = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - (x^2 + y)\frac{\partial}{\partial z}$.
   (a) Show that D is involutive.
   (b) Describe the integral submanifolds of D in $\mathbb{R}^3$.

5. Consider the set E over the real projective space RP^n given by
$$E := \sqcup_{x \in RP^n} E_x$$
   where for each point $x = [x_0 : x_1 : \cdots : x_n] \in RP^n$, E_x is the unique line through the point $(x_0, x_1, \cdots, x_n)$ and the origin in $\mathbb{R}^{n+1}$.
   (a) Show that E is naturally a smooth vector bundle over RP^n.
   (b) Show that E is not isomorphic to the product bundle (i.e. the trivial bundle) over RP^n for any $n \geq 1$.

6. Let $n > 0$. Suppose $f : M \to S^n$ is an immersion from a compact closed, connected n-manifold M to the n-sphere S^n. Prove that f is a diffeomorphism.

7. Consider the noncompact surface $S = \{(x, y, z) : z = x^2 + y^2\} \subset \mathbb{R}^3$.

   (a) Find the supremum for the Gauss curvature and the subset of S on which it is attained.

   (b) Does the Gauss curvature attain its infimum on S? (Explain why or why not!)

8. Prove that the set of upper triangular real 3×3 matrices with determinant 1 is a Lie group. Furthermore,

   (a) How many connected components does this group have?

   (b) Determine its Lie algebra and compute its dimension.