Department of Mathematics and Statistics
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ADVANCED EXAM — DIFFERENTIAL EQUATIONS
AUGUST 2013

Do five of the following seven problems. All problems carry equal weight. Passing level: 75% with at least three substantially complete solutions.

1. Let A be an $n \times n$ matrix whose eigenvalues λ_i satisfy $\text{Re}\ \lambda_i < 0$, for all i. Consider the initial value problem for $x = x(t) \in \mathbb{R}^n$,

$$\begin{aligned} \dot{x} &= Ax + h(x,t)\,, \\ x(0) &= x_0 \in \mathbb{R}^n\,, \end{aligned}$$

where $h(x,t)$ is a smooth function on $\mathbb{R}^n \times [0,\infty)$ taking values in $\mathbb{R}^n$, and $|h(x,t)| \le C|x|^2$ for all x and t.

   (a) Use the variation of parameters method to express the solutions $x(t)$ of this system of ODEs as solutions of an equivalent system of integral equations.

   (b) Use the representation in part (a) to show that, if $|x_0|$ is sufficiently small, then the solution of the IVP is bounded for all t.

2. The ODE system for the Brusselator model of chemical reactants is

$$\dot{x} = 1 - 4x + x^2 y\,, \quad \dot{y} = 3x - x^2 y\,.$$

   (a) Show that the trapezoidal region with the vertices $(1/4, 0)$, $(13, 0)$, $(1, 12)$, $(1/4, 12)$ is a positively invariant set. [Hint: For each side show that the outward normal vector $\mathbf{n}$ satisfies $\mathbf{n} \cdot (\dot{x}, \dot{y}) \le 0$.]

   (b) Find this system's fixed point and determine its type and stability.

   (c) Deduce that this system has a nonconstant periodic solution.

3. (a) State the Poincaré-Bendixson theorem.

   (b) Give an example of an autonomous dynamical system in the plane that has one stable fixed point, one unstable fixed point and one homoclinic orbit.

4. Consider the initial-boundary-value problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = x \qquad (\, 0 < x < 1 \, , \quad t > 0 \,)$$
$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = 0.$$

Exhibit the solution to this IBVP using a Fourier series method.

5. Let $u(x,t)$ by any smooth solution of the following PDE:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0 \, . \tag{1}$$

Assume that the solution exists in all of space ($-\infty < x < +\infty$) for all $t \geq 0$, and that it vanishes rapidly as $|x|$ goes to infinity.

(a) Show that both of the integrals

$$I = \frac{1}{2} \int_{-\infty}^{+\infty} u(x,t)^2 \, dx \qquad J = \frac{1}{2} \int_{-\infty}^{+\infty} \left[\frac{\partial u}{\partial x}(x,t) \right]^2 \, dx \, ,$$

are constant.

(b) State a uniqueness theorem for solutions to the associated initial value problem

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = f(x,t) \, , \quad u(x,0) = \phi(x) \, ,$$

and use part (a) to prove your statement.

6. Let $u(x,y)$ be a solution of the boundary value problem

$$-\Delta u = f(x) \text{ for } x \in \Omega \, , \quad u = \phi(x) \text{ for } x \in \partial\Omega \, ,$$

where $\Omega = B_R(0)$ is the ball of radius $R > 0$ in $\mathbb{R}^3$. Assume that $f \in C^1(\Omega \cup \partial\Omega)$ and $\phi \in C^1(\partial\Omega)$, and let $M_f = \max\{|f(x)| : x \in \Omega\}$ and $M_\phi = \max\{|\phi(x)| : x \in \partial\Omega\}$. Use a maximum principle argument to prove that the following inequality holds

$$|u(x)| \leq M_\phi + M_f \frac{R^2 - x^2 - y^2 - z^2}{6} \quad \text{for all } x \in \Omega \, .$$

7. Consider the so-called Robin boundary value problem

$$\begin{aligned} -\Delta u + \gamma u &= f(x) \quad \text{in } \Omega\,, \\ \frac{\partial u}{\partial \mathbf{n}} + \alpha u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned}$$

for a smoothly bounded domain $\Omega \subset R^n$, with outward unit normal $\mathbf{n}$ on $\partial\Omega$.

(a) Introduce the appropriate Sobolev space for weak solutions u, and explain how the weak form of the BVP is derived from the classical PDE and its boundary conditions.

(b) Given that both the coefficients α and γ are positive constants, prove that this BVP has a unique weak solution for any data $f \in L^2(\Omega)$.

[Hint: Appeal either to the Lax-Milgram theorem or to a variational principle.]