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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $I\!R$. This measure is denoted by m.
3. If a σ-algebra on $I\!R$ is not specified, use the Borel σ-algebra.

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Define the concept that f is convex on $\mathbb{R}$.

   (b) Let μ be a Borel measure on $\mathbb{R}$ such that $\displaystyle\int_{\mathbb{R}} e^{tx} \mu(dx) < \infty$ for all $t \in \mathbb{R}$. Prove that

$$c(t) \;=\; \log \int_{\mathbb{R}} e^{tx} \mu(dx)$$

   is a convex function on $\mathbb{R}$.
   Hint: Use Hölder's inequality.

   (c) Prove that $c'(t)$ exists and that

$$c'(t) \;=\; \frac{\int_{\mathbb{R}} x e^{tx} \mu(dx)}{\int_{\mathbb{R}} e^{tx} \mu(dx)} \,.$$

   Justify all the steps carefully.

2. (a) Let $f \in L^1(\mathbb{R}, m)$ be an integrable function, and define $f_h(x) = f(x-h)$ for $h \in \mathbb{R}$. Show that

$$\lim_{h \to 0} \|f_h - f\|_{L^1} = 0 \,.$$

(b) Give an example of a sequence $\{f_n\}$ such that $f_n \in L^1([0,1], m)$ and

$$\lim_{n \to \infty} \|f_n - f\|_{L^1} = 0$$

but $f_n(x)$ converges for no $x \in [0,1]$.

Hint. Let $f_n = 1_{A_n}$ for appropriate sets A_n.

3. Let $C_{\text{per}}(I\!\!R)$ denote the Banach space of bounded, continuous, real-valued functions on $I\!\!R$ which are periodic of period 2 with the norm $\|f\| = \sup_{|x|\leq 1} |f(x)|$.

For $n \in I\!\!N$ let k_n be a non-negative function in $C_{\text{per}}(I\!\!R)$, and for $g \in C_{\text{per}}(I\!\!R)$ define

$$S_n g(x) \;=\; \int_{-1}^{1} k_n(y) g(x+y)\, dy\,.$$

(a) Prove that S_n defines a bounded linear operator from $C_{\text{per}}(I\!\!R)$ into $C_{\text{per}}(I\!\!R)$.

(b) Assume that for every $n \in I\!\!N$ we have $\int_{-1}^{1} k_n(y)dy = 1$ and that for each $\delta > 0$

$$\lim_{n\to\infty} \sup_{\delta\leq|y|\leq 1} k_n(y) \;=\; 0$$

Prove that

$$\lim_{n\to\infty} \|S_n - \mathrm{I}\| \;=\; 0\,,$$

where I is the identity operator; i.e., $\mathrm{I}f = f$ for $f \in C_{\text{per}}(I\!\!R)$.

4. Compute the following Lebesgue-Stieljes integral

$$\int_{[-2,2]} x^2 dF(x)\,,$$

where

$$F(x) \;=\; \begin{cases} x+2 & \text{if} \quad -2 \leq x \leq -1 \\ 2 & \text{if} \quad -1 < x < 0 \\ x^2+5 & \text{if} \quad 0 \leq x \leq 2 \end{cases}$$

5. Let H be a real, inner product space with inner product $\langle \cdot, \cdot \rangle$, and define $\|x\| = \sqrt{\langle x, x \rangle}$.

(a) Prove the Cauchy-Schwartz inequality relating $x, y \in H$.

(b) Prove that $\|x\|$ defines a norm.

(c) Show that equality holds in the Cauchy-Schwartz inequality if and only if $ax + by = 0$ for some $a, b \in \mathbb{R}$.

6. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|x\|$. Define $L(H, H)$ to be the set of bounded linear operators mapping H into H. Let T be an operator in $L(H, H)$.

   (a) Show that there exists a unique operator $T^* \in L(H, H)$ such that

   $$\langle Tx, y \rangle = \langle y, T^* x \rangle$$

   for all $x, y \in H$.
   Hint: Use the Riesz representation theorem.

   (b) Show the following formula for the operator norm $\|T\|$ of an operator $T \in L(H, H)$:

   $$\|T\| \; = \sup_{\|x\|=1, \|y\|=1} |\langle Tx, y \rangle| \; .$$

   (c) Use part (b) to prove that $\|T\| = \|T^*\|$.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space, and for $1 \leq p \leq \infty$ consider the Banach spaces $L^p(X, \mu)$ and $L^q(X, \mu)$, where q is the conjugate exponent to p; i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

   (a) Prove that for $1 < p < \infty$ and $f \in L^p(X, \mu)$ we have

$$\|f\|_{L^p} = \sup_{\|g\|_{L^q}=1} \left| \int fg d\mu \right| .$$

   (b) Let $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$ two σ-finite measure spaces, and let $f(x_1, x_2)$ be a measurable, non-negative function on $X_1 \times X_2$. Prove that for $1 < p < \infty$

$$\left\| \int f(x_1, x_2) d\mu_2(x_2)) \right\|_{L^p(X_1, \mu_1)} \leq \int \|f(x_1, x_2)\|_{L^p(X_1, \mu_1)} d\mu_2(x_2) .$$

   This inequality is known as the Minkowski inequality for integrals.
   Hint: Use part (a) and Hölder's inequality.

8. For $-\infty < a < b < \infty$ let $f_n : (a, b) \to \mathbb{R}$ be a sequence of functions, each of which is a monotonically increasing function; i.e., if $x < y$, then $f_n(x) \leq f_n(y)$. Suppose that f_n converges to some f almost everywhere with respect to Lebesgue measure. Show that f_n converges to f at all points where f is continuous.
Hint. Approximate any point of continuity of f by sequences lying in the set $\{f_n \to f\}$.