Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

Wednesday August 29, 2012

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.
Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

For any topological spaces X and Y, we denote the set of continuous functions from X to Y by $C(X, Y)$.

(1) Prove that a metric space has a countable dense subset if and only if it has a countable basis for its topology.

(2) Let X be a metric space, and let $f: X \to \mathbb{R}$ be a continuous function and $g_n: X \to \mathbb{R}$ a sequence of continuous functions. Suppose that g_n converges to f pointwise, i.e. $g_n(x) \to f(x)$ for all $x \in X$.
  (a) Show that if X is compact and the sequence g_n is monotonically decreasing (i.e. $g_n(x) \leq g_m(x)$ for any $n \geq m$ and any $x \in X$), then g_n converges to f uniformly.
  (b) Give two examples to show that the compactness and monotone assumptions cannot be removed from this result.

(3) Suppose that X and Y are connected subspaces, and $A \subset X$ and $B \subset Y$ are subspaces with $A \neq X$ and $B \neq Y$. Show that

$$(X \times Y) \smallsetminus (A \times B)$$

is connected.

(4) Given a space X, define an equivalence relation on it by putting $x \sim y$ if for every continous map $f: X \to \{0, 1\}$ (with the discrete topology on $\{0, 1\}$), we have $f(x) = f(y)$. The corresponding equivalence classes are called the *quasi-components* of X.
  (a) Show that each connected component is contained in a quasi-component.
  (b) Let $Y = \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Show that for the space

$$X = (\mathbb{Q} \times \{0\}) \cup (\mathbb{R} \times Y) \subset \mathbb{R} \times \mathbb{R}$$

  the connected components and the quasi-components are different.

(5) Let A and B be locally compact subspaces of a space X.
  (a) Show that $A \cap B$ is locally compact.
  (b) Give an example to show that $A \cup B$ need not be locally compact.

(turn over)

(6) (a) Prove that the one-point compactification $\hat{\mathbb{R}}$ of the real line $\mathbb{R}$ is homeomorphic to the circle S^1.

(b) State and prove a necessary and sufficient condition for a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ to extend to a continuous function $\hat{f}\colon \hat{\mathbb{R}} \to \hat{\mathbb{R}}$.

(7) Let X be the quotient of $D^2 \times \{0, 1\}$ by the equivalence relation generated by $(z, 0) \sim (w, 1)$ if and only if $|z| = |w| = 1$ and $z^4 = w^6$. Compute the fundamental group of X.