BASIC EXAM – ADVANCED CALCULUS & LINEAR ALGEBRA
DEPARTMENT OF MATHEMATICS & STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

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Provide solutions for **seven** of the following ten problems. Each problem is worth 10 points. To pass at the Master's level, it is sufficient to have 42 points (60%), with 3 essentially correct solutions (including at least one from each part); 53 points (75%) with at least two essentially complete solutions from each part is sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded. Be sure to show all your work.

Part I. Linear Algebra

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(1) Let A be the symmetric matrix
$$\begin{pmatrix} 9 & -\sqrt{3} \\ -\sqrt{3} & 11 \end{pmatrix}.$$
Write $A = QDQ^{-1}$ where D is a diagonal matrix and Q is an orthogonal matrix.

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(2) Suppose A is a complex $m \times k$ matrix and B is a complex $k \times n$ matrix. Prove that
$$\operatorname{rank}(AB) \geq \operatorname{rank}(A) + \operatorname{rank}(B) - k.$$

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(3) For a square matrix m, let $\operatorname{ch}_M(x)$ be the characteristic polynomial of M. Suppose A, B are $n \times n$ matrices over the field of complex numbers $\mathbb{C}$. Show that $\operatorname{ch}_{AB}(x) = \operatorname{ch}_{BA}(x)$.
Hint: One approach is first do the case where one of the matrices is invertible, then reduce to the first case via "deformation."

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(4) (a) Let V denote the vector space of real $n \times n$ matrices. Prove that the form
$$\langle A, B\rangle := \operatorname{trace}(A^{\mathrm{t}}B)$$
where A^{t} is the transpose of A, defines an inner product on V. Namely, prove that the form is bilinear, symmetric, and positive definite.
(b) For $n = 2$, find an orthonormal basis of the subspace spanned by
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

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(5) Let T, U be linear transformations of a complex vector space V of dimension n. Show that if $TU = UT$ then T and U have a common eigenvector.

Part II. Advanced Calculus

(6) Find the Taylor series for $f(x) = \frac{1}{2+x}$ and find the interval of convergence of the series.

(7) The unit 2-sphere S^2 in $\mathbb{R}^3$ is the set of points for which $x_1^2 + x_2^2 + x_3^2 = 1$. This is a compact set in $\mathbb{R}^3$. An important result states that every real-valued continuous function on a compact set has both a maximum and minimum.

Now let A be a nonzero symmetric 3×3-matrix over the real numbers. Define the real-valued function

$$f(\mathbf{x}) = \frac{1}{2} A\mathbf{x} \cdot \mathbf{x}$$

for $\mathbf{x} \in \mathbb{R}^3$.

Show, using methods of multivariable calculus, that there exists $\mathbf{v} \in S^2$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero $\lambda \in \mathbb{R}$.

(This proves the existence of an eigenvector of A with nonzero and real eigenvalue; this result generalizes to dimension n).

(8) Compute the line integral $\int_C (x + y)\, dx + y\, dy$ where C in the unit circle with center $(0, 0)$ in two ways:

(a) Directly, by parametrizing C.

(b) Using Green's theorem.

(9) At which points is the function

$$f(x) = \begin{cases} x, & \text{if } x \text{ is irrational} \\ p \sin(\frac{1}{q}), & \text{if } x = \frac{p}{q} \text{ is reduced} \end{cases}$$

continuous?

(10) Evaluate the surface integral $\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (1, 1, z(x^2 + y^2)^2)$ and S is the surface of the cylinder given by $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1$.