DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
ADVANCED QUALIFYING EXAM - DIFFERENTIAL EQUATIONS

Monday, August 27th, 2012

Do five of the following seven problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions, including one from the ODE part (Questions 1-3) and one from the PDE part (Questions 4-7). Please write your work clearly justifying when necessary what you use.

(1) Consider the matrix

$$A = \frac{1}{9}\begin{pmatrix} 5 & 4 & 2 \\ -2 & 11 & 1 \\ -4 & 4 & 11 \end{pmatrix},$$

all of whose eigenvalues are 1.

(a) Find the semisimple-nilpotent decomposition $A = S + N$ of A.

(b) Use this to calculate the matrix exponential and give the general solution of the system

$$\frac{dy}{dt} = A\,y.$$

(2) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, and you are given a non-negative C^1 function $L : \mathbb{R}^n \to \mathbb{R}$ satisfying $L(x) \to \infty$ as $\|x\| \to \infty$, and with the property

$$\langle \nabla L(x), f(x) \rangle \leq c_1 + c_2\, L(x) \quad \text{for all} \quad x,$$

where c_i are non-negative constants. If $x(t)$ solves the ODE

$$\dot{x} = f(x), \quad \text{with} \quad x(t_0) = x_0,$$

find an integral inequality for $L(x(t))$ and use it to show that $L(x(t))$ remains bounded for all t. Conclude that $\|x\|$ remains finite and thus the ODE has a globally defined solution.

(3) Consider the nonlinear system

$$x' = -\lambda(r)\, x + \omega(r)\, y$$
$$y' = -\omega(r)\, x - \lambda(r)\, y$$

where $r = \sqrt{x^2 + y^2}$ and λ and ω are given smooth functions of $r \geq 0$.

(a) Determine the stability of the rest point $(x, y) = (0, 0)$ in terms of $\lambda(0)$ and $\omega(0)$ and describe the qualitative behavior near the origin. Assume that $\omega(0) \neq 0$ and that $\frac{d\lambda}{dr}(0) \neq 0$ if $\lambda(0) = 0$.

(b) Suppose now that

$$\lambda(r) = r\,(1 - r)\,(2 - r) \quad \text{and} \quad \omega(r) = (\frac{1}{2} - r)\,(\frac{3}{2} - r).$$

Describe all periodic orbits and limit sets of the system, and sketch the phase plane.

(4) (a) Determine the type (elliptic, parabolic or hyperbolic) of the equation

$$u_{xx} + 5u_{xy} + 6u_{yy} = 0.$$

Find the characteristic curves, reduce to canonical form and find the general solution.

(b) Use the method of characteristics to find the solution $u = u(x, y)$ of

$$x^2 u_x + xy\, u_y = u^2, \quad u(y^2, y) = 1.$$

Determine whether and where the solution becomes singular.

(5) Prove that

$$K(|x|) \;=\; -\frac{1}{4\pi}\frac{\cos(k|x|)}{|x|}, \qquad\qquad |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

is a **fundamental solution** for $\Delta + k^2$ on $\mathbb{R}^3$.

Hints: If $\phi \in C_0^\infty(\mathbb{R}^3)$ then for large enough R and small $\varepsilon > 0$, apply Green's identity on the set $\Omega_\varepsilon = B_R(0) \setminus B_\varepsilon(0)$. Argue that on the interior sphere $|x| = \varepsilon$, we have

$$\frac{\partial K}{\partial \mathbf{n}} = -\left.\frac{dK}{dr}\right|_{r=\varepsilon} = -\frac{1}{4\pi\varepsilon}(ksin(k\varepsilon) + \frac{\cos(k\varepsilon)}{\varepsilon}).$$

You may use without proof that $K(|x|) = -\frac{\cos(k|x|)}{4\pi|x|}$ is integrable at $x = 0$.

(6) Let $S := \{(x,y) : -1 \le x, y \le 1\}$ be the unit square, and $f : S \to \mathbb{R}$ be a smooth function on S. Prove that any smooth solution $u(x,y,t)$ on $S \times [0,\infty)$ of the equation

$$\begin{cases} u_t = \Delta u + uu_x + uu_y & \text{in} \quad S \times (0,\infty) \\ u(x,y,0) = f(x,y) & \text{for all} \quad (x,t) \in S \end{cases}$$

satisfies the **weak maximum principle**:

$$\max_{S\times[0,T]} u(x,y,t) \le \max\{ \max_{0\le t\le T} u(\pm 1, \pm 1, t),\ \max_{(x,y)\in S} f(x,y) \}$$

for any fixed $T > 0$.

Hint: Consider $u = v + \varepsilon t$ for $\varepsilon > 0$ and analyze the various cases for v to have a maximum.

(7) Let $\Omega \subset \mathbb{R}^n$, $T > 0$ and $u = u(x,t)$ be a smooth solution to the following initial boundary value problem

$$\begin{cases} u_{tt} - \Delta u + u^3 = 0 & \text{in} \quad \Omega \times [0,T] \\ u(x,t) = 0 & \text{for all} \quad (x,t) \in \partial\Omega \times [0,T] \end{cases}$$

(a) Derive an **energy equality** for u.

(b) Show that if $u(x,0) = 0 = u_t(x,0)$ for $x \in \Omega$, then $u \equiv 0$.