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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.
3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of measurable subsets of X.

   (a) Define $\limsup\limits_{n\to\infty} A_n$ and $\liminf\limits_{n\to\infty} A_n$ in terms of appropriate unions and intersections.

   (b) Prove that $x \in \liminf\limits_{n\to\infty} A_n$ if and only if x lies in all but finitely many A_n and that $x \in \limsup\limits_{n\to\infty} A_n$ if and only if x lies in infinitely many A_n.

   (c) Prove that $\mu(\liminf\limits_{n\to\infty} A_n) \leq \liminf\limits_{n\to\infty} \mu(A_n)$.

   (d) Prove that if $\sum\limits_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(\limsup\limits_{n\to\infty} A_n) = 0$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, $\{f_n\}_{n\in\mathbb{N}}$ a sequence of measurable functions taking values in $[0, 1]$, and f a measurable function taking values in $[0, 1]$. We say that f_n converges to f in distribution if

$$\lim_{n\to\infty} \int_X g \circ f_n \, d\mu = \int_X g \circ f d\mu$$

for all continuous functions $g : [0, 1] \to \mathbb{R}$.

(a) State the definition that "f_n converges to f in measure."

(b) Show that if f_n converges to f in measure, then f_n converges to f in distribution.

3. Let μ_F and μ_G be finite Borel measures on $\mathbb{R}$ with the respective distribution functions F and G; i.e., they are the unique Borel measures such that for any interval $(a,b]$

$$\mu_F((a,b]) = F(b) - F(a)\,, \quad \mu_G((a,b]) = G(b) - G(a)\,.$$

Let $\mu_F \star \mu_G$ denote the Borel measure with distribution function given by the convolution

$$F \star G(x) = \int_{-\infty}^{\infty} F(x-y)\,\mu_G(dy).$$

(a) Prove that for any Borel set B

$$\mu_F \star \mu_G(B) = \int_{\mathbb{R}} \mu_F(B-y)\,\mu_G(dy),$$

where $B - y = \{x \in \mathbb{R} : x = b - y \text{ for some } b \in B\}$ is the translate of the set B.

(b) Prove that if g is a nonnegative Borel measurable function on $\mathbb{R}$, then

$$\int_{\mathbb{R}} g(x)\,\mu_F \star \mu_G(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(x+y)\,\mu_F(dx)\,\mu_G(dy).$$

4. Let $[a, b]$ be a finite interval and $F : [a, b] \to \mathbb{R}$ a function.

   (a) State the definition that "F is of bounded variation."

   (b) Show that if F is increasing and bounded, then F is of bounded variation.

   (c) Show that F is of bounded variation if and only if $F = G_1 - G_2$, where G_1 and G_2 are increasing and bounded.

5. For each $j = 1, 2$, let $(X_j, \mathcal{M}_j)$ be a measurable space and let μ_j and ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$.

   (a) For $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$ define

$$\alpha(E) = \int_E \left(\frac{d\nu_1}{d\mu_1}(x_1) \cdot \frac{d\nu_2}{d\mu_2}(x_2) \right) d(\mu_1 \times \mu_2)(x_1, x_2).$$

   Prove that α is a measure on $\mathcal{M}_1 \otimes \mathcal{M}_2$ and that $\alpha = \nu_1 \times \nu_2$.

   (b) Prove that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and that

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \cdot \frac{d\nu_2}{d\mu_2}(x_2).$$

6. Let $(X, \mathcal{M}, \mu)$ be a measure space with μ a σ-finite measure and let $f \in L^1(\mu)$ be a nonnegative function.

(a) Show that

$$(\mu \times m)\Big\{(x, y) \in X \times \mathbb{R} \,:\, 0 \leq f(x) \leq y\Big\} \;=\; \int_X f d\mu;$$

i.e., the integral of f equals the area under the graph of f.

(b) Show that

$$(\mu \times m)\Big\{(x, y) \in X \times \mathbb{R} \,:\, f(x) = y\Big\} \;=\; 0;$$

i.e., the measure of the graph of f equals 0.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space and $0 < p < \infty$. Define

$$L^p(\mu) \;=\; \left\{ f : X \to \mathbb{C} : f \text{ measurable} \;,\; \int_X |f|^p \, d\mu < \infty \right\}$$

and, as usual, identify two functions f and g if $f = g$ μ-almost everywhere.

(a) Show that if $1 \leq p < \infty$, then $\|f\|_p \equiv (\int_X |f|^p \, d\mu)^{1/p}$ defines a norm on $L^p(\mu)$.
Hint: Use Hölder's inequality.

(b) Show that if $0 < p \leq 1$, then $d(f, g) \;=\; \int_X |f - g|^p \, d\mu$ defines a metric on $L^p(\mu)$.

(c) Show that if $1 < p < \infty$, then $d(f, g) \;=\; \int_X |f - g|^p \, d\mu$ does <u>not</u> define a metric on $L^p(\mu)$.

8. Suppose that $\mathcal{H}$ is an infinite-dimensional Hilbert space.

   (a) Show that there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ with $\|f_n\| = 1$ such that $\{f_n\}$ has no convergent subsequence.

   (b) Show that for any sequence $\{f_n\}_{n\in\mathbb{N}}$ with $\|f_n\| = 1$ there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and $f \in \mathcal{H}$ such that

   $$\lim_{k\to\infty} \langle f_{n_k}\,,\, g\rangle \;=\; \langle f\,,\, g\rangle$$

   for all $g \in \mathcal{H}$.
   Hint: Let g run through a basis of $\mathcal{H}$ and use a diagonalization argument. Express f in terms of the basis.