DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ADVANCED EXAM — ALGEBRA

Passing Standard: It is sufficient to do FIVE problems correctly, including at least ONE FROM EACH of the THREE parts.

Part I. Group Theory and Representation Theory

1. Show that there is no simple group of order 120.

2. Let G be a non-Abelian group of order 8.
(a) Show that G has a unique irreducible representation ρ of degree > 1.
(b) Determine every non-Abelian group G of order 8 such that ρ in part (a) satisfies $\det(\rho(g)) = 1$ for all $g \in G$.

3. Consider the group $G = \langle a, b, c : \ a^2 = b^3 = c^5 = abc \rangle$.
(a) Show that the element abc lies in the center of G.
(b) Show that the quotient group $G/\langle abc \rangle$ is a finite group.

Part II. Commutative Algebra

4. Let J be an ideal of $\mathbf{Z}[x]$. Set $n := \min\{\deg(f) : f \in J\}$. If J contains a monic polynomial of degree n, show that J is a principal ideal.

5. Let R with a commutative ring with 1. Show that every R-module is projective if and only if every R-module is injective.

6. Let $A \subset B$ be an integral ring extension. Suppose $B \setminus A$ is multiplicatively closed in B. Show that A is integrally closed in B.

Part III. Field Theory and Galois Theory

7. Let F_1, F_2 be intermediate subfields of a finite extension K/k. Prove that the k-algebra $F_1 \otimes_k F_2$ is a field if and only if $[F_1F_2 : k] = [F_1 : k][F_2 : k]$.

8. Show that there are infinitely many irreducible polynomials over any field.

9. Compute Galois group of $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over $\mathbb{Q}$.