DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
AUGUST 2011

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correcct

PhD: 75% with at least three substantially correct.

1. We wish to find the smallest positive root x^* of
$$f(x) = 1 + \cos x.$$
   (a) What is the order of convergence of Newton's method for x^*?
   (b) To possibly accelerate convergence, we use a modified Newton's method,
$$x_{n+1} = x_n - C\frac{f(x_n)}{f'(x_n)}.$$
   Find the value of C for which this iteration converges fastest, and determine the order of convergence.

2. Consider real square matrices only.
   (a) Show that a matrix is non-singular if it is positive definite.
   (b) Find a matrix which is non-singular and non- positive-definite. Justify your answer.
   (c) Is a positive-definite matrix always symmetric? If yes, prove it. Otherwise, find a positive definite matrix which is not symmetric. (Be sure to show your example is indeed positive definite.)

3. Find the quadrature
$$\int_{-1}^{1} f(x)\ dx \approx \omega_1 f(-1) + \omega_2 f(x_1)$$
   with the highest possible degree of precision.

4. Solve the ODE $dy/dx = f(x, y)$ with a scheme as
$$y_{n+1} \quad = \quad y_n + hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n))$$
   Find the leading term of the truncation error. What is the order of convergence?

5. Suppose that equispaced points (including the two boundary points) are used in the polynomial interpolation of

$$f(x) = \begin{cases} 1, & x \in [-1, 0], \\ -1, & x \in (0, 1]. \end{cases}$$

   (a) Find the interpolation polynomials of degree one and two (denoted as $p_1(x)$ and $p_2(x)$))

   (b) Compute the L_∞ (or *maximum*) errors of these three interpolations.

   (c) Is the polynomial interpolation on equispaced points convergent for $f(x)$ in terms of the *maximum* error? Why or why not?

6. Consider a set $\{x_0, x_1, \ldots, x_n\}$ of $n+1$ *distinct* points, and the corresponding Lagrange basis functions $\{\ell_0(x), \ell_1(x), \ldots, \ell_n(x)\}$, where

$$\ell_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)} \qquad \rightarrow \qquad \ell_k(x_j) = \delta_{kj}\,.$$

   Prove that

$$\sum_{k=0}^{n} \ell_k(x) = 1\,.$$

7. An $n \times n$ complex matrix A is said to be *normal* if it commutes with its conjugate transpose, i.e. $AA^* = A^*A$, where $A^* = \bar{A}^T$.

   (a) Schur's Theorem states that any complex matrix is unitarily similar to an upper triangular matrix. Use Schur's theorem to prove the following *Spectral Theorem for Normal Matrices* for a given matrix A:

$$A \text{ is normal} \iff A \text{ is unitarily similar to a diagonal matrix}$$

   (b) Given *any* square matrix A, it can be shown that

$$\lim_{k\to\infty} A^k = 0 \iff \rho(A) < 1,$$

   where $\rho(A)$ is the *spectral radius* of A.

   Prove this result when A is normal.