DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM—APPLIED MATHEMATICS
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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. Consider the 2×2 linear system

$$\frac{dx}{dt} = Ax\,, \qquad \text{with} \quad A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}.$$

(a) Find the general solution of this system.
(b) Calculate the exponential matrix solution e^{tA}.

2. Consider a mechanical system with configuration variable $q = q(t) \in R^1$ governed by the differential equation

$$\frac{d^2q}{dt^2} + 2\beta\frac{dq}{dt} + q - q^2 \;=\; 0\,, \qquad \text{for a constant} \quad 0 < \beta < 1\,.$$

(a) Find the equilibrium points, and classify them by type and stability.
(b) Draw the phase portrait in the $(q, dq/dt)$ plane, and describe the qualitative behavior of this system.

3. Consider the wedge-shaped region in the plane R^2 given in polar coordinates by $\Omega = \{\,(r,\theta)\,:\,0 < r < a\,,\;0 < \theta < \beta\,\}$, for given radial extent a and opening angle β. Solve Laplace's equation inside Ω subject to the boundary conditions:

$$u = 0 \quad \text{for } \theta = 0 \text{ and } \theta = \beta, \qquad \frac{\partial u}{\partial r} = h(\theta) \quad \text{for } r = a\,.$$

The boundary data $h(\theta)$ is any continuous function with $h(0) = 0 = h(\beta)$.

4. Consider the inhomogeneous heat equation with unit diffusivity on an interval of length π having insulated boundary conditions at each end. That is, consider the following initial-boundary value problem for the temperature $u = u(x,t)$:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t) \quad \text{in} \quad 0 < x < \pi, \;\; t > 0\,,$$

$$u(x,0) = 0\,, \qquad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(\pi,t)\,.$$

The source term f is an arbitrary smooth function compatible with the initial and boundary conditions.

(a) Solve this problem by the Fourier series method.

(b) Express the solution in the integral form

$$u(x,t) = \int_0^t \int_0^\pi G(t-t', x, x')\, f(x',t')\, dx'\, dt'\ ,$$

and give a formula for G.

5. Consider the differential equation:

$$\frac{dx}{dt} = x\left[1 - (x-\alpha)^2\right] \qquad \text{for} \quad \alpha > 0.$$

(a) Determine the equilibrium points for $\alpha < 1$ and $\alpha > 1$, and describe the stability of each.

(b) What kind of bifurcation occurs at $\alpha = 1$?

6. **(a)** Find the steady state solution $u(x,t) = v(x)\sin\omega t$ to the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad 0 < x < L,$$

with $u(0,t) = 0$ and $u(L,t) = A\sin\omega t$. Assume that $\omega/c \neq m\pi/L$ for any $m = 1, 2, \ldots$.

(b) What happens when $\omega/c = m\pi/L$ for some $m = 1, 2, \ldots$.?