University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
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Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. Suppose that N_1 and N_2 are codimension one embedded submanifolds of a manifold M. Suppose further that N_1 and N_2 intersect transversely, i.e. for all $p \in N_1 \cap N_2$, the tangent spaces T_pN_1 and T_pN_2 span T_pM. Then $N_1 \cap N_2$ is a codimension 2 submanifold of M.

   (a) Show that there exists a Riemannian metric g on M so that for every point $p \in N_1 \cap N_2$, normal vectors at p to N_1 and N_2 are orthogonal. In particular any normal vector to N_1 is tangent to N_2 and vice-versa.

   (b) Show that if M, N_1, and N_2 are all orientable, then so is $N_1 \cap N_2$.

2. Let M be a connected n-dimensional manifold, and let $p, q \in M$ be distinct points. Show that the deRham cohomology $H^{n-1}_{dR}(M \setminus \{p, q\})$ is not zero.

3. Let ω be a symplectic 2-form on $\mathbb{R}^2$, i.e., ω is closed and non-degenerate. Show that for any point $p \in \mathbb{R}^2$, there is a local coordinate system (x, y) near p such that $\omega = dx \wedge dy$.

   Hint:

   step 1: show that near p, $\omega = d\alpha$ for some nonvanishing 1-form α.

   step 2: show that $\alpha = f dg$ for some smooth functions f, g near p.

   step 3: use the non-degeneracy of ω to finish the proof.

4. Give a proof or a counterexample for each statement.

   (a) If ω is an $(n-1)$-form on a compact smooth n-manifold M (without boundary), then there exists $p \in M$ so that $d\omega(p) = 0$.

   (b) If α is a non-vanishing 1-form on a manifold M, then there exists a 1-form β so that $\alpha \wedge \beta$ is non-vanishing.

5. Let (M, g) be an oriented Riemannian manifold.

   (a) Define the divergence $\operatorname{div} X$, of a smooth vector field X on M.

   (b) Let θ_t be the flow generated by the vector field X. (if you like, you can assume X is complete.) Show that

   $$\frac{d}{dt}\theta_t^*(dV_g)\bigg|_{t=0} = (\operatorname{div} X)dV_g$$

   where dV_g is the Riemannian volume form. (Hint: one approach is to prove it first in a neighborhood of a point where $X_p \neq 0$; this means you can put X in a simple form. Then use continuity to handle the points where $X_p = 0$.)

   (c) Show that if $\operatorname{div} X = 0$, then for any compactly supported $f \in C^\infty(M)$, the integral $\int_M (f \circ \theta_t)dV_g$ is independent of t.

6. Let J be the $n \times n$ matrix with k entries 1 and $n-k$ entries -1 on the diagonal and zeros everywhere else. Show that

   $$G = \{A \in GL(n, \mathbb{R}) \mid AJA^t = J\}$$

   is a Lie subgroup of $GL(n, \mathbb{R})$. Compute its dimension and its Lie algebra as a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

7. Let M be the surface in $\mathbb{R}^3$

   $$M = \{(r\cos\theta, r\sin\theta, r) \mid r, \theta \in \mathbb{R},\ r > 0\}.$$

   (a) Write the differential equations for parallel transport of a tangent vector $X(t) = f(t)\partial_r + g(t)\partial_\theta$ around the loop $r(t) = 1$, $\theta(t) = t$, $0 \le t \le 2\pi$.

   (b) Show that the Gaussian curvature of M is identically zero. Why does this not contradict part (a)?

8. For any $0 < a \le 1$, let T_a be the quotient of $\mathbb{R}^2$ by the equivalence relation generated by $(x, y) \sim (x + a, y)$ and $(x, y) \sim (x, x + 1/a)$ for all $(x, y) \in \mathbb{R}^2$.

   (a) Show that T_a is a compact orientable smooth manifold and the standard metric $dx^2 + dy^2$ on $\mathbb{R}^2$ induces a metric g_a on T_a.

   (b) Show that the manifolds T_a are all diffeomorphic, and they all have the same total volume (i.e., the integral of the volume form is independent of a).

   (c) Show that T_a and T_b are not isometric unless $a = b$. Hint: look at closed geodesics.