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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.
3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. (a) Let $A \subset \mathbb{R}$ be an arbitrary subset of the real line (not necessarily Lebesgue measurable) and let $m^*(A)$ denote the exterior (or outer) measure of A. Show that there exists a Lebesgue measurable set $B \subset \mathbb{R}$ such that $A \subset B$ and $m(B) = m^*(A)$.

   (b) Prove that the Lebesgue exterior (or outer) measure is continuous from below. In other words, if $\{A_n\}_{n \geq} 1$ is increasing sequence of sets, i.e., $A_1 \subset A_2 \subset \cdots \subset \mathbb{R}$, prove that

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} m^*(A_n)\,.$$

   Hint: Use part (a). You may use that m is continuous from below.

2. Suppose $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, 3, \cdots$ is a sequence of measurable functions such that f_n converges to f for every $x \in \mathbb{R}$. Show that f is a measurable function.

3. (a) Let $f : [a, b] \to \mathbb{R}$ be Lebesgue integrable. Show the absolute continuity of the integral, i.e., show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable set $A \subset [a, b]$ with $m(A) \le \delta$ we have $|\int_A f dm| \le \varepsilon$.

   (b) Let $f : [a, b] \to \mathbb{R}$ be Lebesgue integrable and let $F : [a, b] \to \mathbb{R}$ be the function given by

$$F(x) = \int_{[a,x]} f dm \,.$$

   Show that F is continuous and of bounded variation.

4. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and let $T : \mathcal{X} \to \mathcal{X}$ be a linear map. The map T is called *bounded* if T maps the bounded sets of $\mathcal{X}$ into bounded sets in $\mathcal{X}$.

   (a) Prove that the linear map T is bounded if and only if there exists a constant $C > 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in \mathcal{X}$

   (b) Prove that the linear map T is bounded if and only if T is continuous.

5. Let $([0,1], \mathcal{B})$ be the unit interval with the Borel σ-algebra. Let $M([0,1])$ be the space of real finite measures $\mu : \mathcal{B} \to \mathbb{R}$ with the norm $\|\mu\| = |\mu|([0,1])$. The space $M([0,1])$ is a normed vector space (you do not need to prove this). Prove that $M([0,1])$ is a Banach space.

6. Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing, right-continuous function, and let $\Phi : \mathbb{R} \to \mathbb{R}$ be a continuous increasing invertible function. Let μ_F and $\mu_{F\circ\Phi}$ be the Lebesgue-Stieljes measures associated to F and $F \circ \Phi$ respectively. Show that if $f \in L^1(\mu_F)$, then $f \circ \Phi \in L^1(\mu_{F\circ\Phi})$ and

$$\int f d\mu_F = \int f \circ \phi d\mu_{F\circ\Phi} .$$

Hint: It is enough to consider non-negative f and to prove the inequality $\int f \circ \Phi d\mu_{F\circ\Phi} \le \int f d\mu_F$ (why?).

7. Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x,y) = \begin{cases} 2 & \text{if } 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Let μ be the measure on $\mathbb{R}^2$ which is absolutely continuous with respect to the Lebesgue measure $m \times m$ on $\mathbb{R}^2$ with with Radon Nikodym derivative

$$\frac{d\mu}{d(m \times m)} = g\,.$$

Let $T : \mathbb{R}^2 \to \mathbb{R}$ be the map given by $T(x,y) = x$ and let $\tau = \mu \circ T^{-1}$ be the measure on $\mathbb{R}$ given by

$$\tau(A) = \mu(T^{-1}(A))\,.$$

Find the Lebesgue decomposition of the Lebesgue measure m on $\mathbb{R}$ with respect to τ, $m = m_{ac} + m_{sing}$ and compute the Radon-Nykodym derivative $\frac{dm_{ac}}{d\tau}$.

8. A function $f : [a, b] \to \mathbb{R}$ is convex if for all $x, y \in [a, b]$ and $0 \le t \le 1$ we have

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)\,,$$

i.e., the graph of f lies below every one of its chords.

(a) Show that the secant lines move monotonely. In other words, prove that the slope of the secant line, i.e. the function

$$f_t(x) = \frac{f(x+t) - f(x)}{t}\,,$$

is an increasing function of t and of x.

(b) A function $f : [a, b] \to \mathbb{R}$ is Lipschitz continuous if there exists a constant L such that for all $x, y \in [a, b]$ we have $|f(x) - f(y)| \le L|x - y|$. Show that if f is Lipschitz continuous then f is absolutely continuous.

(c) By the result in (a), the left and right derivatives of a convex function f exist for all x and agree outside a countable set (you **do not** need to prove this). Show that f is convex if and only if f is absolutely continuous and $f'(x)$ is increasing.

Hint: For the if part use the fundamental theorem of calculus for $f_t(x)$.