Department of Mathematics and Statistics
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Basic Exam: Topology

Wednesday, September 1, 2010

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.
Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

For any topological spaces X and Y, we denote the set of continuous functions from X to Y by $C(X, Y)$.

(1) (a) Let $f, g : X \to Y$ be two continuous maps where Y is Hausdorff and X is any topological space. Let $A \subset X$ be dense. Show that if f and g agree on A, then $f = g$.

(b) Let Z be Hausdorff and $A \subset Z$. Let $i\colon A \to Z$ be the inclusion. Suppose there exists a continuous map $r\colon Z \to A$ such that $r(i(a)) = a$ for all $a \in A$. (r is called a retraction.) Prove that A must be closed (hint: you can use part (a)).

(2) Show that a compact metric space has a finite basis for its topology.

(3) Find a subspace $X \subset \mathbb{R} \times \mathbb{Z}$ for which (1) the intersection of X with $\mathbb{R} \times \{i\}$ is a closed bounded interval for every $i \in \mathbb{Z}$, and (2) the restriction of the projection $p_1\colon \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ to X is surjective, but is *not* a quotient map.

(4) Let $f : X \to Y$ be a continuous map. Suppose that f is closed and that $f^{-1}(p)$ is compact for every $p \in Y$. Prove that $f^{-1}(K)$ is compact for any compact subset $K \subset Y$.

(5) Let X be the space $[0, 1]^\omega$ of sequences with values in $[0, 1]$, taken with the box topology. Show that X is not path connected. What is the path component containing the sequence $(0, 0, 0, \dots)$?

(6) Let $U_1 \subset U_2 \subset U_3 \subset \dots$ be a nested sequence of open subsets of $\mathbb{R}^n$, with the standard (Euclidean) topology. Suppose that $0 \in U_1$, each U_i is connected, and that the homomorphism $\pi_1(U_i, 0) \to \pi_1(U_{i+1}, 0)$ of fundamental groups induced by the inclusion of U_i in U_{i+1} is trivial for any $i \geq 1$. If $U = \bigcup_i U_i$, show that $\pi_1(U, 0)$ is trivial.

(turn over)

(7) Let $X = C([0,1],[0,1])$ be the set of continuous functions from the unit interval to itself. Define a function $M\colon X \to [0,1]$ by

$$M(f) = \sup_{x \in [0,1]} f(x).$$

Show that M is continuous if X is given the uniform topology, but not if X is given the point-open topology (topology of pointwise convergence).